TOPIC 9

Practice with eigenstuff

Suppose we have a transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$.

Then we are either in one of the following situations or the other:

Non-singular case

- $\det(f) \neq 0$
- $\ker(f) = \{\mathbf{0}\}$
- $\operatorname{ran}(f) = \mathbb{R}^n$
- f is invertible
- All eigenvalues of f are nonzero
- The vectors $f({}_1), \ldots, f({}_n)$ are linearly independent (and thus form a basis for $\mathbb{R}^n$).
- The matrix
  $$\begin{pmatrix} | & & | \\ f({}_1) & \ldots & f({}_n) \\ | & & | \end{pmatrix}$$
  row reduces to the identity.
- The equation $f(\mathbf{v}) = \mathbf{r}$ has exactly one solution for each vector $\mathbf{r}$.

Singular case

- $\det(f) = 0$
- $\ker(f) \neq \{\mathbf{0}\}$
- $\operatorname{ran}(f) \neq \mathbb{R}^n$
- f is not invertible
- $\lambda = 0$ is an eigenvalue of f
- The vectors $f({}_1), \ldots, f({}_n)$ are linearly dependent (and thus do not form a basis for $\mathbb{R}^n$).
- The matrix
  $$\begin{pmatrix} | & & | \\ f({}_1) & \ldots & f({}_n) \\ | & & | \end{pmatrix}$$
  does not row reduce to the identity.
- For each vector $\mathbf{r}$, the equation $f(\mathbf{v}) = \mathbf{r}$ either has no solution or has many solutions.

Exercise 9.1. Compute the eigenvalues and eigenspaces of the transformation

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x + 2y \\ x + 2y + 3z \end{pmatrix}$$

Exercise 9.2. Compute the eigenvalues and eigenspaces of the transformation

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ x + 2y + z \\ x + 3z \end{pmatrix}$$

Exercise 9.3. Compute the eigenvalues and eigenspaces of the transformation

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y - z \\ -x + 2y - z \\ -x = y + 2z \end{pmatrix}$$

Exercise 9.4. Suppose $\lambda = 0$ is one of the eigenvalues of transformation f. What can we say about the kernel of f? What can we say about the determinant of f? What can we say about the invertibility of f?

Exercise 9.5. Suppose we know that none of the eigenvalues of transformation f are zero. What can we say about the kernel of f? What can we say about the determinant of f? What can we say about the invertibility of f?

Exercise 9.6. Find the eigenvalues and eigenspaces of

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Exercise 9.7. Find the eigenvalues and eigenspaces of

$$\begin{pmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{pmatrix}$$

Exercise 9.8. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an invertible transformation having eigenvalues $\lambda_1, \ldots, \lambda_n$.

(1) What (if anything) can be said about the eigenvalues & eigenvectors of f^{-1}?
(2) What (if anything) can be said about the eigenvalues & eigenvectors of f^2? (Here f^2 is the transformation that does f twice.)

Exercise 9.9. Construct an example of an invertible 2×2 matrix which has no real eigenvalues. Interpret the corresponding transformation geometrically.