TOPIC 8

Determinants

Basic setup

- Have a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$
- We build the associated matrix

$$M = \begin{pmatrix} | & | & & | \\ f(\mathbf{e}_1) & f(\mathbf{e}_2) & \dots & f(\mathbf{e}_n) \\ | & | & & | \end{pmatrix}$$

- We want to compute a number $\Delta = \det(M)$ such that $\delta \neq 0$ corresponds to f being invertible

Two dimensional case

- $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- $\det(M) = ad - bc$
- Geometrically we can interpret this as the *signed area* of the parallelogram determined by the vectors

$$f(\mathbf{e}_1) = \begin{pmatrix} a \\ c \end{pmatrix} \qquad f(\mathbf{e}_2) = \begin{pmatrix} b \\ d \end{pmatrix}$$

- Picture appears in Figure 1

  Area of the parallelogram is

$$\begin{aligned} \Delta &= 2\left[\frac{1}{2}(a+b)(c+d) - \frac{1}{2}ac - \frac{1}{2}bd - bc\right] \\ &= ad - bc \end{aligned}$$

- determinant is positive if $\dfrac{a}{c} > \dfrac{b}{d}$; thus it represents *signed area*
- If we swap the order of the vectors, the sign of Δ changes

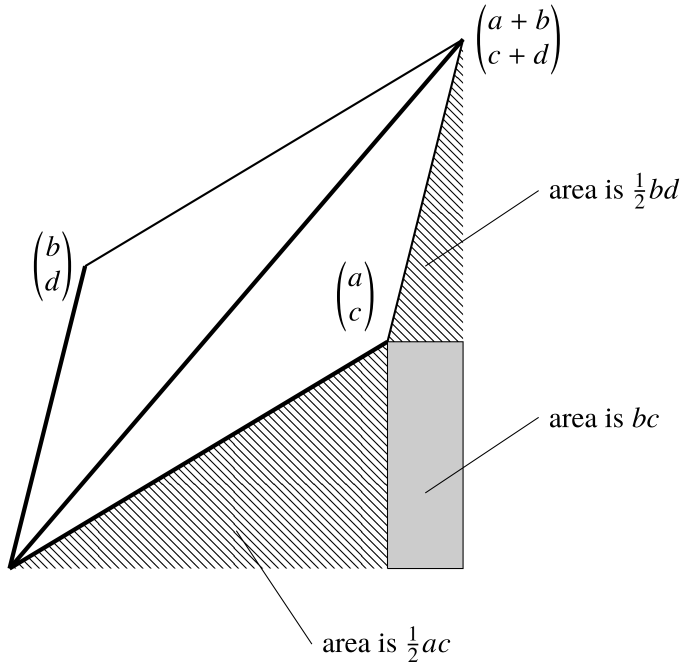

FIGURE 1. The area of the parallelogram can be computed by subtracting the various shaded regions from the area of the triangle. . . and then doubling the result.

- If we swap the roles of x and y (i.e., swap the rows of the matrix), the sign of Δ changes
- This motivates us to define Δ in higher dimensions to be the "volume" of the parallelepiped defined by the column vectors of the matrix.
- Geometrically, the transformation f takes the "unit parallelepiped" (having volume 1) to a parallelepiped of volume $|\Delta|$, and where the orientation has been reversed if $\Delta < 0$.

How to compute $\Delta = \det(M)$ in higher dimensions?

- First method: understand the geometric implications of row reduction

  Row swaps: We've already seen that swapping adjacent rows changes Δ by a factor of -1

  Row scalings: Multiplying a row by α has the effect of rescaling that direction by α. Thus this has the effect of multiplying Δ by α.

  Row addition: Replacing (row i) by (row i)+α(row j) has the geometric effect of "sliding" the parallelotope in such a way that the volume doesn't change; thus this doesn't change Δ—see Figure 2, and also the homework.

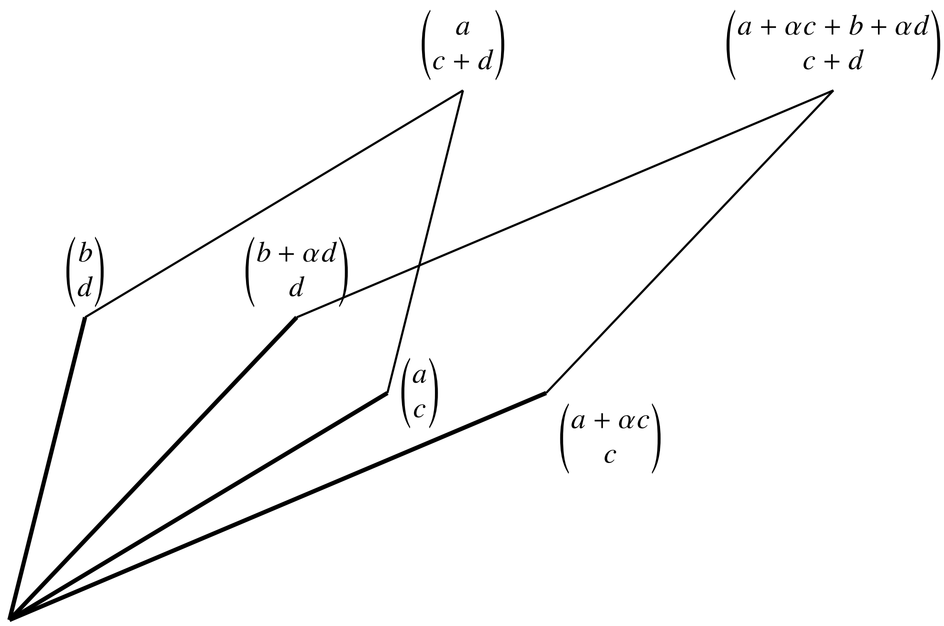

FIGURE 2. The area of the parallelogram is preserved by the row operation.

- Example:

$$\begin{array}{cc} \text{matrix} & \text{determinant} \\ \begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix} & \Delta \\ \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 3 & 9 \end{pmatrix} & \frac{1}{2}\Delta \\ \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} & \frac{1}{2}\Delta \\ \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} & \frac{1}{6}\Delta \\ \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \frac{1}{6}\Delta \end{array}$$

Thus $1 = \frac{1}{6}\Delta$ and $\Delta = 6$. This agrees with our old formula!

- We can do this for larger matrices, too . . . in fact from a numerical analysis point of view this is a very fast algorithm. [Jeff says that anyone who has taken CS 171 needs to code this up.]
- Notation: We denote that we are taking the determinant with vertical bars. Thus if
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
then we use the notation
$$\det(M) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
It is important to remember that even though these vertical bars look like absolute value signs, it is possible for the determinant to be negative!

There exist several other formulas that are useful for computing determinants, and that are useful for theoretical reasons.

Write
$$M = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} v_1^1 & v_2^1 & & v_n^1 \\ v_1^2 & v_2^2 & \dots & v_n^2 \\ \vdots & \vdots & & \vdots \\ v_1^n & v_2^n & \dots & v_n^n \end{pmatrix}$$
so that v_i^j is the j^{th} entry of $\mathbf{v}_i$.

- Cofactor formula
  - For entry v_i^j let the *cofactor* C_i^j be $(-1)^{i+j}$ times the determinant of the matrix obtained by removing column i and row j from M. (This matrix is called the *minor* of v_i^j.)
  - The determinant can be obtained by summing over either a row or a column the products of an entry times its cofactor:
$$\begin{aligned} \det(M) &= v_i^1 C_i^1 + v_i^2 C_i^2 + \dots v_i^n C_i^n \qquad \text{for any fixed } i \\ &= v_1^j C_1^j + v_2^j C_2^j + \cdots + v_n^j C_n^j \qquad \text{for any fixed } j \end{aligned}$$
- Combinatorial formula
  - Let σ be a permutation of n items, with the sign $\operatorname{sgn}(\sigma)$ equal to 1 if the permutation can be accomplished with an even number of adjacent transpositions and -1 if the permutation can be accomplished with an odd number of adjacent transpositions.

– Then

$$\det *M) = \sum_{\sigma} \operatorname{sgn}(\sigma)\, a_1^{\sigma(1)} \cdot a_2^{\sigma(2)} \cdot \ldots \cdot a_n^{\sigma(n)}$$
$$= \sum_{\sigma} \operatorname{sgn}(\sigma)\, a_{\sigma(1)}^{1} \cdot a_{\sigma(2)}^{2} \cdot \ldots \cdot a_{\sigma(n)}^{n} \qquad (1)$$

– For details, pick up any linear algebra book. . . or check out the internet

Exercise 8.1. Compute (by hand, showing your work/reasoning) the determinant of the following matrices:

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then figure out how to compute determinants using *Mathematica* or another appropriate technology. Use this technology to check your computations.

Exercise 8.2. Compute the following determinants:

$$\begin{vmatrix} 7 & 1 & 10 \\ 17 & 1 & 20 \\ 27 & 1 & 30 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 3 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 & 1 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{vmatrix}$$

Exercise 8.3. Suppose we know that

$$\begin{vmatrix} a & b & c \\ p & q & r \\ t & u & v \end{vmatrix} = 2.$$

Compute

$$\begin{vmatrix} 2t+p & 2u+q & 2v+r \\ p-a & q-b & r-c \\ \frac{1}{4}a & \frac{1}{4}b & \frac{1}{4}c \end{vmatrix}.$$

Exercise 8.4. Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-b)(c-a)(b-a).$$

Exercise 8.5. Suppose A is a square matrix, and that one of the rows of A is a multiple of another row. What do you know about $\det(A)$? Explain your reasoning.

Exercise 8.6. Suppose A is a square matrix, and that one of the columns of A is a multiple of another column. What do you know about $\det(A)$? Explain your reasoning.

Exercise 8.7. A $n \times n$ matrix is called *upper triangular* if all the entries "below the diagonal" are zero. For example, a 4×4 upper triangular matrix takes the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

where $*$ represents various numbers. Show that the determinant of an upper triangular matrix is simply the sum of the diagonal entries.

Exercise 8.8. Let's explore the "sliding" procedure illustrated in Figure 2.

(1) Show that the "sliding" of the parallelogram can be described by the transformation $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ having formula

$$R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \alpha y \\ y \end{pmatrix}.$$

(2) Show that the matrix for R has determinant 1.

(3) Explain why we can conclude that R doesn't change areas (only reshapes them), and thus that the two parallelograms have the same area.

Let's now generalize this idea that we can describe row operations using transformations. We continue to work with the transformation $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that has associated matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(4) Find a transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that applying S to the column vectors of M has the effect of scaling the first row by a factor of α. Show that the determinant of the transformation S is α and thus that S scales volumes by a factor of α.

(5) Find a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that applying T to the column vectors of M has the effect of exchanging the two rows. Show that the determinant of the transformation T is -1 and thus that T changes the sign of the signed volume.

Finally, let's consider what happens for transformations in higher dimensions, and suppose we have a transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and that the associated matrix is M.

(6) Can you find a transformation $R\colon \mathbb{R}^n \to \mathbb{R}^n$ that, when applied to the column vectors of M, has the effect of adding α(row j) to (row i)? What is the determinant of R?

(7) Can you find a transformation $S\colon \mathbb{R}^n \to \mathbb{R}^n$ that, when applied to the column vectors of M, has the effect of scaling (row i) by a factor of α? What is the determinant of S?

(8) Can you find a transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ that, when applied to the column vectors of M, has the effect of swapping (row i) and (row j)? What is the determinant of T?