TOPIC 7

Properties of linear transformations $\mathbb{R}^n \to \mathbb{R}^n$

If a transformation takes a space to itself we can ask special questions

- Understand geometry of transformation in special way
- The possibility of a "reversible" transformation (Rank-Nullity theorem)

The eigenvalue problem

- Idea: Suppose we have a transformation. . . we would like to find a basis that "best captures its geometric behavior."
- First step: Look for linear subspaces that get stretched. Fancy name: the eigenvalue problem
  Note: Only makes sense for transformations $\mathbb{R}^n \to \mathbb{R}^n$
- Formal statement of the eigenvalue problem: Find a vector $\mathbf{v}$ and a constant λ such that $f(\mathbf{v}) = \lambda\mathbf{v}$.
- Geometric interpretation: If $\mathbf{v}$ and λ solve the problem, then the transformation f stretches span$\{\mathbf{v}\}$ by a factor of λ
- Example: $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9x + 3y \\ 3x + y \end{pmatrix}$
- Easy way to compute: $\mathbf{v}$ and λ solve eigenvalue problem for f exactly when $\mathbf{v} \in \ker(f - \lambda\,\mathrm{id})$.
- When does the transformation $f\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ have a nontrivial kernel?
  - Row reduce to discover

$$ad - bc = 0 \qquad \leftrightarrow \qquad \ker(f) \neq \{\mathbf{0}\}$$
$$ad - bc \neq 0 \qquad \leftrightarrow \qquad \ker(f) = \{\mathbf{0}\}$$

  - quantity $\Delta = ad - bc$ called the *determinant* of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

  - We'll call this the *matrix associated to f*
  - Soon we'll generalize this to $n \times n$ matrices...
- Example: $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 4y \\ 4x - 5y \end{pmatrix}$

Inverses of transformations

- Consider $f : \mathbb{R}^n \to \mathbb{R}^n$ (same dimension)
- If $\ker(f) = \{\mathbf{0}\}$, then there exists $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that $f \circ f^{-1}$ and $f^{-1} \circ f$ are the *identity transformation* $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$
- There's a trick for computing... row reduce the "double matrix"
- Examples: $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9x + 3y \\ 3x + y \end{pmatrix}$ and $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 4y \\ 4x - 5y \end{pmatrix}$

Exercise 7.1. Suppose $\lambda = 0$ is one of the eigenvalues of linear transformation f. Explain why $\ker(f)$ is not trivial.

Exercise 7.2. Suppose $\mathbf{v}$ and λ are a solution to the eigenvalue problem for transformation f. Explain why any vector $\mathbf{w}$ in $\mathrm{span}\{\mathbf{v}\}$ is also an eigenvector of f with eigenvalue λ.

Exercise 7.3. Suppose λ is an eigenvalue of transformation f. Explain why the collection of associated eigenvectors forms a subspace; this subspace is called the *eigenspace associated to λ*.

Exercise 7.4. Consider the linear transformation $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x + 4y \\ 4x - 2y \end{pmatrix}$.

(1) Find the eigenvalues of f. For each eigenvalue find a basis for the associated eigenspace.
(2) Compute the determinant of the matrix associated to f.
(3) Find a formula for the inverse transformation f^{-1}.

Exercise 7.5. Consider the linear transformation $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 4y \\ 4x - 3y \end{pmatrix}$.

(1) Find the eigenvalues of f. For each eigenvalue find a basis for the associated eigenspace.
(2) Compute the determinant of the matrix associated to f.
(3) Find a formula for the inverse transformation f^{-1}.