TOPIC 6

Properties of linear transformations $\mathbb{R}^n \to \mathbb{R}^m$

Using bases to understand transformations

- Suppose we have a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and we know $f(\mathbf{b}_1), \ldots, f(\mathbf{b}_n)$. The linearity properties of f tell us how $\mathbf{v} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$ transforms.
- Example: Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is such that

  $$f(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad f(\mathbf{e}_2) = \begin{pmatrix} 3 \\ -7 \end{pmatrix}.$$

  Then

  $$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 3y \\ 2x - 7y \end{pmatrix}.$$

- Example: Consider the basis $\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$. Suppose we have a transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that

  $$f(\mathbf{b}_1) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad f(\mathbf{b}_2) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

  What is the general formula for f?

- Construct a transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that stretches the $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ direction by a factor of 5, and that rotates the positive x axis to the negative y axis.

Rank-Nullity Theorem

- Formal statement: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\dim(\ker(f)) + \dim(\operatorname{ran}(f)) = n$.
- Why is this true?

- Build a basis of $\mathbb{R}^n$ of the form $\{\mathbf{k}_1, \ldots, \mathbf{k}_q, \mathbf{b}_1, \ldots, \mathbf{b}_p\}$, where $\{\mathbf{k}_1, \ldots, \mathbf{k}_q\}$ is a basis for $\ker(f)$.
- The vectors $f(\mathbf{b}_1), \ldots, f(\mathbf{b}_p)$ are linearly independent (why?) and thus form a basis for $\operatorname{ran}(f)$.

Exercise 6.1. Suppose $\mathbf{v}$ and $\mathbf{w}$ are in the kernel of some linear transformation f and that α is a real number. Show that the properties of f imply that $\alpha\mathbf{v}$ and $\mathbf{v} + \mathbf{w}$ are also in $\ker(f)$. Explain how this shows that $\ker(f)$ is a linear subspace.

Exercise 6.2. Suppose a linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$ has a *trivial* kernel (meaning the kernel is $\{\mathbf{0}\}$). Describe the range of the transformation geometrically.

Exercise 6.3. Suppose that the range of a linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$ is a line. What can you say about the kernel of the transformation?

Exercise 6.4. Construct a transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that stretches the subspace span $\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$ by a factor of 7 and that has kernel spanned by $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Exercise 6.5. Construct a transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors clockwise by an angle of $\frac{\pi}{3}$ and stretches them by a factor of 4.

Exercise 6.6. Construct a transformation $\mathbb{R}^2 \to \mathbb{R}^3$ such that the range space of the transformation is the plane described by $2x - 3y + 4z = 0$.

Exercise 6.7. Construct a transformation $\mathbb{R}^3 \to \mathbb{R}^2$ such that the kernel of the transformation is the plane $x - y + z = 0$ and such that the image of the transformation is the line $2x + 3y = 0$.