TOPIC 5

Linear transformations of $\mathbb{R}^n$

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if

- (f respects scaling) $f(\alpha \mathbf{v}) = \alpha f(\mathbf{v})$ for all vectors $\mathbf{v}$ and real numbers α
- (f respects addition) $f(\mathbf{v} + \mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w})$ for all vectors $\mathbf{v}, \mathbf{w}$

Consequences of definition

- Subspaces are mapped to subspaces

Example:

- $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 3x - 2y \end{pmatrix}$
- Where is $\{\mathbf{0}\}$ mapped to?
- Where are various lines mapped to? (start with the axes. . .)
- Where is $\mathbb{R}^2$ mapped to?

```
f[x_, y_] := {2 x + 3 y, 3 x - 2 y};
Show[
 ParametricPlot[{{t, 0}, {0, t}}, {t, 0, 1},
  PlotStyle -> {{Thickness[0.01], Red}, {Thickness[0.01], Blue}},
  PlotRange -> 1.5, Frame -> True],
 ParametricPlot[{x, y}, {x, 0, 1}, {y, 0, 1}, Mesh -> {4, 4},
    Axes -> False]
 ]
Show[
 ParametricPlot[{f[t, 0], f[0, t]}, {t, 0, 1},
  PlotStyle -> {{Thickness[0.01], Red}, {Thickness[0.01], Blue}},
  PlotRange -> 1.5, Frame -> True],
 ParametricPlot[f[x, y], {x, 0, 1}, {y, 0, 1}, Mesh -> {4, 4},
    Axes -> False]
 ]
```

- Since image is all of $\mathbb{R}^2$, the system of equations

$$2x + 3y = a$$
$$3x - 2y = b$$

has a solution for any numbers a, b

Another example:

- $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 2x + 3y \end{pmatrix}$
- all vectors taken to the line span $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$
- The corresponding system of equations only has a solution if RHS is on the line

Fun fact: Linear transformations maps subspaces to subspaces

Four important subspaces associated to a linear transformation f:

domain: the subspace of inputs
codomain: the subspace where the outputs all live
kernel $\ker(f)$**:** the subspace inside the domain consisting of all vectors mapped to zero
range $\operatorname{ran}(f)$**:** the subspace inside the codomain consisting of all outputs of f

Notes:

- The word "range" has a slightly different meaning from in calculus class!
- the range is sometimes called the *image*; an alternate symbol is $\operatorname{im}(f)$.

Examples:

(1) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 3x - 2y \end{pmatrix}$
- domain is $\mathbb{R}^2$
- codomain is $\mathbb{R}^2$
- $\ker(f) = \{\mathbf{0}\}$
- $\operatorname{ran}(f) = \mathbb{R}^2$

(2) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 2x + 3y \end{pmatrix}$
- domain is $\mathbb{R}^2$
- codomain is $\mathbb{R}^2$
- $\ker(f) = \operatorname{span}\left\{\begin{pmatrix} 3 \\ -2 \end{pmatrix}\right\}$
- $\operatorname{ran}(f) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

Subspaces of a transformation and solutions to linear systems:

- Write the system as $f(\mathbf{v}) = \mathbf{r}$
- System has solutions if and only if $\mathbf{r} \in \operatorname{ran}(f)$
- If there are solutions, there are as many solutions are there are elements in the kernel:
  - If $\ker(f) = \{\mathbf{0}\}$, then solutions are unique
  - If $\ker(f) \neq \{\mathbf{0}\}$, then there are multiple solutions, and they take the form
  $$\mathbf{v} = \mathbf{w} + \mathbf{k},$$
  where $\mathbf{w}$ is some fixed *particular solution* and $\mathbf{k} \in \ker(f)$.

Examples

(1) Since range is all of $\mathbb{R}^2$, there is always a solution. Since $\ker(f) = \{\mathbf{0}\}$, the solution is unique.

(2) There is only a solution if $\mathbf{r} \in \operatorname{span}\left\{\begin{pmatrix} 3 \\ -2 \end{pmatrix}\right\}$. If there is a solution, then it takes the form
$$\mathbf{v} = \mathbf{w} + \alpha \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$
For example, if $\mathbf{r} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$...

Examples–for each do the following:

- identify domain, codomain
- identify kernel, range; describe in terms of a basis
- find all solutions to an associated system of equations

Tip: To understand range, look at image of standard basis vectors...

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y + 5z \\ 2x - 3y - z \end{pmatrix}, \qquad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 2x - 3y \\ x + y \end{pmatrix}, \qquad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y + 5z \\ 2x - 3y - z \\ x - y \end{pmatrix}$$

Exercise 5.1. Consider the transformation
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \\ x + y \end{pmatrix}$$

(1) What is the domain of f?
(2) What is the codomain of f?
(3) What is the kernel of f? Describe the kernel in terms of a basis. What is the dimension of $\ker(f)$?
(4) What is the range of f? Describe the range in terms of a basis. What is the dimension of $\operatorname{ran}(f)$?
(5) Find all solutions to the equation

$$f(\mathbf{v}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(6) Find all solutions to the equation

$$f(\mathbf{v}) = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$

Exercise 5.2. Consider the transformation

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ x - y + z \end{pmatrix}$$

(1) What is the domain of f?
(2) What is the codomain of f?
(3) What is the kernel of f? Describe the kernel in terms of a basis. What is the dimension of $\ker(f)$?
(4) What is the range of f? Describe the range in terms of a basis. What is the dimension of $\operatorname{ran}(f)$?
(5) Find all solutions to the equation

$$f(\mathbf{v}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(6) Find all solutions to the equation

$$f(\mathbf{v}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Exercise 5.3. Consider the transformation

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + y \\ x - 2y \\ 2x - y \end{pmatrix}$$

(1) What is the domain of f?
(2) What is the codomain of f?
(3) What is the kernel of f? Describe the kernel in terms of a basis. What is the dimension of $\ker(f)$?
(4) What is the range of f? Describe the range in terms of a basis. What is the dimension of $\operatorname{ran}(f)$?
(5) Find all solutions to the equation

$$f(\mathbf{v}) = \begin{pmatrix} 4 \\ 5 \\ 0 \\ 3 \end{pmatrix}$$

(6) Find all solutions to the equation

$$f(\mathbf{v}) = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 5 \end{pmatrix}$$

Exercise 5.4. Consider the transformation

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 3y - 2z \\ 2x + y + 3z \\ x - y \end{pmatrix}$$

(1) What is the domain of f?
(2) What is the codomain of f?
(3) What is the kernel of f? Describe the kernel in terms of a basis. What is the dimension of $\ker(f)$?
(4) What is the range of f? Describe the range in terms of a basis. What is the dimension of $\operatorname{ran}(f)$?

(5) Find all solutions to the equation

$$f(\mathbf{v}) = \begin{pmatrix} 0 \\ -5 \\ -4 \end{pmatrix}$$

(6) Find all solutions to the equation

$$f(\mathbf{v}) = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$