TOPIC 4

Subspaces of $\mathbb{R}^n$

The Cartesian space $\mathbb{R}^n$

- Generalizes $\mathbb{R}^2$, $\mathbb{R}^3$
- Typically we think in terms of vectors, rather than points

$$(x^1, \ldots, x^n) \leftrightarrow \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$$

- Sub-/super-script convention:
  - A list of vectors has subscripts: $\mathbf{v}_1, \mathbf{v}_2, \ldots$
  - The entries in a vector have superscripts
  - If we have a list of vectors, then v_3^5 is the 5th entry of the 3rd vector.
  - Warning: Other notation will eventually arise
- The standard basis $\mathcal{E} = \mathcal{B}_{\text{std}} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

Linear subspaces V of $\mathbb{R}^n$

- hyperplanes: generalizations of lines, planes
- described by equations, just as before
- here is where matrix notation is really nice

Exercise 4.1. Consider the system

$$\begin{aligned} x - y + 2z + w &= 0 \\ x - z - w &= 0 \\ 3x - y - z &= 0. \end{aligned}$$

(1) Describe (geometrically) the set of solutions, viewed as a subspace of $\mathbb{R}^4$.
(2) What is a basis for the solution subspace? What is the dimension of the solution subspace?

Exercise 4.2. Determine whether the following vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}$$

are linearly independent.

Exercise 4.3. Repeat Exercise 4.2 with the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ -2 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 3 \\ 3 \\ -3 \end{pmatrix}.$$

Exercise 4.4. Determine if the following vectors are linearly independent:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 5 \\ 5 \end{pmatrix} \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 7 \\ -7 \end{pmatrix}$$

Exercise 4.5. Suppose we have a system of m equations in n unknowns, and that the reduced row echelon form has p pivots. What is the dimension of the space of solutions?