TOPIC 3

Linear independence

Linear independence for a list of vectors

- Formalizes the notion of "not redundant" — want to rule out "round trip" combinations
- Technical definition: A collection of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is *linearly independent* if the only solution to

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k = 0$$

  is $\alpha_1 = 0, \ldots, \alpha_k = 0$.
- We can express condition as system of equations. . . or using matrix.

Linear independence and RREF

- The condition $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k = 0$ is equivalent to

$$\left[\begin{array}{ccc|c} | & & | & 0 \\ \mathbf{v}_1 & \cdots & \mathbf{v}_k & \vdots \\ | & & | & 0 \end{array}\right]$$

  Note: Since right side is all zeroes, we can ignore/suppress it.
- RREF is identity matrix $\leftrightarrow$ $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are linearly independent
- RREF is not identity matrix $\leftrightarrow$ $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are linearly dependent

Bases for linear subpaces

- Suppose we are given a list of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$.
- the *span* of the list is all vectors of the form $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k$
- Denote this $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$
- Formal definition: A collection of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a *basis* for V if
  - the collection $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, and
  - if $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = V$.
- The *dimension* of V is the number of vectors in any basis.

Exercise 3.1. Give a geometric description of the following subspaces.

(1) span $\left\{\begin{pmatrix}2\\3\end{pmatrix}\right\}$

(2) span $\left\{\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}-6\\-9\end{pmatrix}\right\}$

(3) span $\left\{\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}2\\-3\end{pmatrix}\right\}$

(4) span $\left\{\begin{pmatrix}2\\3\\4\end{pmatrix},\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$

(5) span $\left\{\begin{pmatrix}2\\3\\4\end{pmatrix},\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\7\\10\end{pmatrix}\right\}$

Exercise 3.2. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix}1\\3\\5\end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix}2\\-2\\3\end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix}-1\\13\\9\end{pmatrix}$$

(1) Determine if the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.
(2) Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find a basis for V. What is the dimension of V?

Exercise 3.3. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix}1\\3\\2\end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix}-2\\-6\\-4\end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix}7\\21\\14\end{pmatrix}$$

(1) Determine if the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.
(2) Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find a basis for V. What is the dimension of V?

Exercise 3.4. In each of the following sentences, one or more vocabulary words is used incorrectly. Please write down a corrected sentence.

(1) The line passing through the points $(0, 0)$ and $(2, 3)$ is a one-dimensional basis.

(2) The vectors $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ form a span of the x,y plane.

(3) A basis must always contain the zero vector.

(4) The span of a plane contains two vectors that do not point in the same direction.

(5) The span of $\begin{pmatrix}1\\3\\1\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}$ is a two-dimensional basis.

(6) A list of linearly independent vectors forms a subspace of their span.

Exercise 3.5. Each of the following sentences is not correct. For each, please (i) provide an example to illustrate that the sentence is false, and (ii) write a corrected sentence.

(1) The span of any two vectors is a plane.

(i) The span of the vectors $\begin{pmatrix}1\\2\end{pmatrix}$ *and* $\begin{pmatrix}2\\4\end{pmatrix}$ *is the line* $y = 2x$*, not a plane.*
(ii) The span of any two vectors that are not colinear is a plane.

(2) A list of vectors is a basis for a subspace if the span of those vectors is the subspace.

(3) If a list of 3D vectors is linearly independent, then that list is a basis for 3D space.

(4) All of the vectors in a subspace must be linearly independent.