TOPIC 1

Linear systems in two and three dimensions

1. Content

Outline.

(1) Systems of linear equations
(2) Gaussian elimination
(3) Systems of equations in augmented matrix form; Gaussian elimination for augmented matrices
(4) Geometry of linear equations
(5) Linear structure for subspaces of $\mathbb{R}^3$: bases (informal, geometric), coordinates, dimension

Mathematica. You can have *Mathematica* solve systems of equations using the `Solve` function.

In order to solve the system

$$2x + 3y = 7$$
$$2x + 4y = 4$$

one uses the following code:

```
Solve[2x+3y==7 && 2x+4y==4, {x,y}]
```

Alternatively, one can first define the system, then solve it:

```
system = 2x+3y==7 && 2x+4y==4;
Solve[system]
```

Mathematica will also do Gaussian elimination in matrix form. Consider the system

$$2x + 3y = 5$$
$$2x + 4z = 8$$
$$x + y + z = 0$$

This is entered in to *Mathematica*, and subsequently reduced, using the following code:

```
matrix = {{2,3,0,5}, {2,0,4,8}, {1,1,1,0}};
MatrixForm[RowReduce[matrix]]
```

Finally, here is a fun demo:

```
plane = Plot3D[(-x - 2 y)/3, {x,-5,5}, {y,-5,5}];
basis1 = ParametricPlot3D[{-2 t, t, 0}, {t,0,1},
   PlotStyle -> {Thickness[.01], Red}];
basis2 = ParametricPlot3D[{-3 t, 0, t}, {t,0,1},
   PlotStyle -> {Thickness[.01], Blue}];
Manipulate[
 Show[plane, basis1, basis2,
  ParametricPlot3D[{t (-2 y - 3 z), t (y), t (z)}, {t,0,1},
   PlotStyle -> {Thickness[.02], Purple}]], {y,-5,5}, {z,-5,5}]
```

2. Homework problems

Exercise 1.1. Find all solutions to the system of equations

$$\begin{aligned} x + 2y - 3z &= 0 \\ 2x - 2y + 3z &= 0 \\ x + y + z &= 5 \end{aligned}$$

Exercise 1.2. Find all solutions to the system of equations

$$\begin{aligned} y - 2z &= 1 \\ 3x + z &= 4 \\ 2x + y - 5z &= 0 \end{aligned}$$

Exercise 1.3. Find all solutions to the system of equations

$$\begin{aligned} 3x - y + z &= 0 \\ 5x - y + 2z &= 0 \end{aligned}$$

Interpret the set of solutions geometrically.

Exercise 1.4. Consider the system of equations

$$\begin{aligned} x + y - 2z &= 0 \\ x - y + z &= 0 \\ 3x + y + \lambda z &= 0 \end{aligned}$$

where λ is some real parameter. For which values of λ does the set of solutions form a line? Give the equation of the corresponding line(s).

Exercise 1.5. Consider the system of equations

$$2x - 3y = 5 \qquad ax + by = c,$$

where a, b, c are real numbers.

(1) Choose values for a, b, c so that the system has only one solution. Make a sketch showing the lines that correspond to each of the equations in the system. Show, on your sketch, the solution to the system.
(2) Choose values for a, b, c so that the system has no solution. Make a sketch showing the lines that correspond to the equations.

Exercise 1.6.

(1) Find an equation for the plane in $\mathbb{R}^3$ which contains the y-axis and the point $(1, 1, 1)$.
(2) Then find an equation for a plane which does not coincide with the previous plane, but which does pass through both the point $(1, 1, 1)$ and the point $(0, 1, 0)$.
(3) Use Gaussian elimination to find an equation for the line passing through both $(1, 1, 1)$ and $(0, 1, 0)$.
(4) What might have been an easier way to construct the equation of the line?

Exercise 1.7. Consider the equation

$$x + 3y - 2z = 0.$$

(1) Solve the equation in the "usual manner."
(2) The set of solutions forms a subspace of $\mathbb{R}^3$. Geometrically, what kind of subspace is it? Find a basis $\mathcal{B}$ for the subspace, and determine the dimension of the subspace.
(3) Since $(x, y, z) = (1, 3, 5)$ solves the equation, it must be that

$$\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

lies in the subspace described above. What are the coordinates of $\mathbf{v}$ with respect to basis $\mathcal{B}$?

Exercise 1.8. Consider the equation

$$x - y + 3z = 0.$$

(1) Argue that the space of solutions describes a subspace that is a plane in $\mathbb{R}^3$.
(2) Solve the equation in the "usual manner" (by viewing x as the dependent variable) and obtain a basis for the plane; call this basis $\mathcal{B}$.
(3) Then solve the equation viewing y as the dependent variable; call the resulting basis C.
(4) Since $(x, y, z) = (1, 4, 1)$ solves the equation, it must be that

$$\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$

lies in the plane. What are the coordinates of $\mathbf{v}$ with respect to basis $\mathcal{B}$? What are the coordinates of $\mathbf{v}$ with respect to basis C?

Exercise 1.9. Consider the system of equations

$$x + y - 2z = 0$$

$$2x - y - z = 0$$

(1) Describe (geometrically) the subspace of solutions.
(2) Find a basis for the subspace and determine the dimension of the space of solutions.