Reviewing for the final exam

Basic concepts: You need to be able to give the *formal definition*, give an *informal description*, and give a *geometric interpretation* of the following concepts:

(1) The *derivative* of a function.

(2) The *definite integral* of a function.

You also need to be able to give informal descriptions of:

(3) The *limit* of a function.

(4) The *general antiderivative* of a function.

You need to know what the following words mean:

(5) a *critical point* of a function

(6) an *inflection point* of a function

(7) a function is *increasing/decreasing*

(8) a function is *concave up/down*

(9) a *maximum/minimum point* of a function.

You also need to know how to find the

(10) *linear approximation* of a function centered at input x_*.

Finally, you need to know how to

(11) compute a limit using *l'Hopital's Rule*.

Computing derivatives: You need to know how to compute simple derivatives using the definition (i.e. using limits). You also need to know how to compute derivatives using the various shortcut rules that we learned.

(1) Compute the derivative of the following functions *using the definition*:

(a) $f(x) = 2x^2 + 3$ (b) $f(x) = \frac{3}{x}$ (c) $f(x) = \sqrt{x}$

(2) Give the derivative of the following.

(a) $\sqrt{x}$
(b) $\frac{1}{x}$
(c) $\sin x$
(d) $\cos x$
(e) $\tan x$
(f) e^x
(g) $\ln x$
(h) $\sin^{-1} x$
(i) $\tan^{-1} x$

(3) Compute the derivative of the following

(a) $f(x) = \dfrac{x^2 - \sqrt{x}}{x^5}$
(b) $g(t) = t^3 \left(5t^2 - 6\sqrt{t}\right)$
(c) $g(t) = \sqrt{t}\ \sin t$
(d) $h(t) = \sqrt{\sin t}$
(e) $h(x) = \cos(\sqrt{x} - 5x)$
(f) $f(x) = \dfrac{\sin x}{x}$
(g) $f(t) = \frac{1}{1+t^2}$
(h) $g(x) = e^{-x} \tan(5x)$
(i) $f(x) = x \tan^{-1}(2x)$
(j) $p(t) = e^{-4t^2}$

Graphical properties of functions: Do the following for each of the functions below:

- Find the roots, critical points, and inflection points.
- Find the regions where the function is increasing/decreasing.
- Find the regions where the function is concave up/down.
- Find the limit as $x \to \pm\infty$ of the function.
- Find the asymptotes of the function (if there are any).
- Make a rough sketch of the graph of the function.

(1) $f(x) = \dfrac{x+2}{2x-3}$

(2) $f(x) = 2x^3 + 3x^2 - 36x$

(3) $f(x) = \dfrac{2}{1+x^2}$

(4) $f(x) = \dfrac{2}{1-x^2}$

Linear approximations:

(1) Find the linear approximation of $f(x) = e^{-4x}$ centered at $x_* = 0$.

(2) Find the linear approximation of $f(x) = 4\ln(x)$ centered at $x_* = 1$.

As a bonus, find the quadratic (second-order) approximations.

Optimization problems:

(1) Suppose we have a right triangle such that the length of the hypotenuse is 10. What should the length of the legs be in order to get the triangle with largest area?

(2) Suppose we have a cylinder such that the combined area of the side, top, and bottom add up to 100. Find the dimensions of the cylinder of largest volume.

(3) What point on the parabola given by $y = x^2 - 6x + 9$ is closes to the point $(0, 0)$?

(4) Suppose we have a species living in a habitat that can only sustainable support a population of 8 (measured in millions). According to the Lotka-Volterra model, the growth rate of the population will be given by $rP(1 - P/8)$, where P is the size of the population at the moment. What size of population corresponds to largest growth rate?

Related rates problems:

(1) Suppose the radius of a circle is increasing at a rate of 8 meters per second. At what rate is the area changing at the moment when the area is 1?

(2) In introductory chemistry class we are taught that the relation between the pressure (P), volume (V), and temperature (T) of one mole of ideal gas are related by the formula $PV = 8.3T$. (Here we use SI units.) Suppose that the temperature is held constant, but the pressure is increasing at a rate of 3 atmospheres per minute. At what rate is the volume changing right at the moment when the volume is 10 cubic meters?

(3) In introductory physics class we are taught that the force (F) on an object, the mass (m) of an object, and the acceleration (a) of an object all satisfy $F = ma$. Suppose that Paul throws a ball of snow at his brother. The ball of snow is falling under the force of gravity, which is $F = -9.8$ kilogram-meters per second^2. The ball of snow is also

coming apart in such a way that it is losing mass at a rate of 0.05 kilograms per second. At what rate is the acceleration changing right at the moment when the mass is .5 kilograms?

Definite integrals and cumulative effect:

(1) Suppose Paul walks with a velocity given by $v(t) = 3t - t^2$. How far does he go between time $t = 1$ and $t = 2$?

(2) Compute these definite integrals:

(a) $\int_0^4 (x^2 - \sqrt{x})\,dx$

(b) $\int_{-1}^1 \frac{1}{1+x^2}\,dx$

(c) $\int_0^5 e^{-x}\,dx$

(d) $\int_{-\pi/2}^{\pi/2} \cos x\,dx$

(e) $\int_{-2}^2 x^7\,dx$

(f) $\int_1^9 \frac{1}{\sqrt{x}}\,dx$