ASSIGNMENT 9

Three important shortcut rules

Reading: Sections 2.4 and 2.5 of Minton–Smith

We have various ways to combine old functions in order to make new functions. Given functions f and g we can

- Add them to make a new function $f(x) + g(x)$
- Multiply them to make a new function $f(x)\,g(x)$
- Compose them to make a new function $f(g(x))$.

Example

- Suppose $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = 3x^2 - 5$
- $f(x) + g(x) = \frac{1}{\sqrt{x}} + 3x^2 - 5$
- $f(x)\,g(x) = \frac{3x^2-5}{\sqrt{x}}$
- $f(g(x)) = \frac{1}{\sqrt{3x^2-5}}$

Question:

- How to relate the derivatives of f and g to the derivatives of these new functions
- We already know the *addition rule*

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$$

How to deal with products

- Recall that $f(x + h) = f(x) + f'(x)\,h + \ldots$ and $g(x + h) = g(x) + g'(x)\,h + \ldots$
- Compute

$$f(x + h)\,g(x + h) = f(x)g(x) + \left[f'(x)\,g(x) + f(x)\,g'(x)\right]\,h + \ldots$$

- Thus we obtain the *product rule*

$$\boxed{\frac{d}{dx}\left[f(x)\,g(x)\right] = \frac{d}{dx}\left[f(x)\right]\,g(x) + f(x)\,\frac{d}{dx}\left[g(x)\right].}$$

- Example

$$\begin{aligned}\frac{d}{dx}\left[\frac{3x^2-5}{\sqrt{x}}\right] &= \frac{d}{dx}\left[\frac{1}{\sqrt{x}}(3x^2-5)\right]\\ &= \frac{d}{dx}\left[\frac{1}{\sqrt{x}}\right](3x^2-5) + \frac{1}{\sqrt{x}}\,\frac{d}{dx}\left[3x^2-5\right]\\ &= \ldots\end{aligned}$$

- Example $\frac{d}{dx}\left[(2x-3)(4x+7)\right] = \ldots$

How to deal with compositions

- We compute

$$\begin{aligned}f(g(x+h)) &= f\left(\underbrace{g(x)}\; + \underbrace{g'(x)\,h + \ldots}\right)\\ &= f(g(x)) + f'\left(g(x)\right)\,g'(x)\,h + \ldots\end{aligned}$$

- Thus we obtain the *chain rule*

$$\boxed{\frac{d}{dx}\left[f(g(x))\right] = f'\left(g(x)\right)\,g'(x)}$$

We can also write this as

$$\boxed{\frac{d}{dx}\left[f(g(x))\right] = f'\left(g(x)\right)\,\frac{d}{dx}\left[g(x)\right]}$$

- Example $\frac{d}{dx}\left[\frac{1}{\sqrt{3x^2-5}}\right] = \ldots$
- Example $\frac{d}{dx}\left[(2x+5)^{12}\right] = \ldots$

Tips for success

- Examine function to understand its structure
- Remember order of operations – work from inside out

Example

$$\frac{d}{dx}\left[\frac{2x^3+4x}{\sqrt{4x^2+9}}\right] = \frac{d}{dx}\left[(2x^3+4x)\frac{1}{\sqrt{(4x^2+9)}}\right] = \ldots$$

Exercise 9.1. Compute the derivative of the following functions.

(1) $f(t) = (t^3 - 3t^2)\sqrt{t+1}$

(2) $f(t) = \dfrac{t^3 - 3t^2}{\sqrt{t+1}}$

(3) $f(t) = \dfrac{1}{\sqrt{1-t^2}}$

(4) $f(t) = \sqrt{1+t^2}$

(5) $f(t) = \dfrac{t}{\sqrt{1-t^2}}$

(6) $f(t) = \dfrac{t^2 - 1}{t^2 + 1}$

Exercise 9.2. Find the linear approximation of the function $f(x) = t\sqrt{1+t}$ centered at $x = 0$.

Exercise 9.3. Find the linear approximation of the function $f(x) = \frac{1}{1+4t^2}$ centered at $x = 0$.

For additional practice, I recommend problems 5–16 in section 2.5 of Minton–Smith.