ASSIGNMENT 8

# Linear approximations

Reading: The first two pages of section 3.1 in Smith–Minton. (Technically this stuff is also in section 2.1, but that notation in that section is weird.)

Approximation problem:

- Given a function $f(t)$, approximate $f$ by an "easy" function near input $t_*$.
- Idea: approximate by a linear function $P_1(t)$

Linear approximation

- We can compute $f(t_*)$ and $f'(t_*)$.
- Start with a function having same rate of change as $f$ does at time $t_*$.
- Adjust so that it is centered at the point $(t_*, f(t_*))$.
- Result is *first order approximation (centered at $t_*$)*

$$P_1(t) = f(t_*) + f'(t_*)\,(t - t_*)$$

- The book refers to these are "tangent lines"

Examples

- Find the first order approximation of $f(t) = \dfrac{1}{\sqrt{t}}$ centered at $t = 9$
- Find the first order approximation of $f(t) = \dfrac{5 - t^2}{\sqrt{t}}$ centered at $t = 4$
- Find the first order approximation of $f(x) = (x + 7)^2$ at $x = 3$

Revisit previous example

- Note that $f(3 + h) = 100 + 20h + h^2 = f(3) + f'(3)\,h + \dots$
- For a general function we have

$$f(x_* + h) = f(x_*) + f'(x_*)\,h + \dots$$

- We can reverse the logic. Compute

$$f(x_* + h) = f(x_*) + \boxed{\phantom{xx}}\,h + \text{stuff that vanishes}\dots$$

  Whatever appears in the box is $f'(x_*)$.

**Exercise 8.1.** Find the first order approximation of the function $f(x) = \sqrt{x}$ centered at $x = 1$

**Exercise 8.2.** Find the first order approximation of the function $g(x) = \frac{1}{x}$ centered at $x = 1$.

**Exercise 8.3.** Find the first order approximation of the function $f(t) = \frac{t^2 - 7\sqrt{t}}{t^3}$ centered at $t = 4$.

**Exercise 8.4.** In this exercise we explore the relationship between first order approximations and multiplication.

(1) Find the first order approximation of the function $h(x) = \frac{1}{\sqrt{x}}$ centered at $x = 1$.

(2) Explain why $h(x) = f(x)\,g(x)$, where $f$ and $g$ are from Exercises 8.1 and 8.2.

(3) Suppose you multiply together the approximations you found in Exercises 8.1 and 8.2. How does this compare to what you found in the part (1)?

**Exercise 8.5.** In this exercise we explore *quadratic approximations*, focusing on the function $f(x) = \sqrt{x}$. The goal is to find a quadratic polynomial $P_2(x)$ that approximations $f$ near $x = 1$.

(1) Show that $f(1) = 1$, $f'(1) = \frac{1}{2}$, and $f''(1) = -\frac{1}{4}$.

(2) Consider now the generic polynomial $P_2(x) = a_0 + a_1 x + a_2 x^2$, where $a_0$, $a_1$, $a_2$ are numbers. Show that $P_2(0) = a_0$, $P_2'(0) = a_1$, and $P_2''(0) = 2a_2$.

(3) Explain why choosing $a_0 = 1$, $a_1 = \frac{1}{2}$, and $a_2 = -\frac{1}{8}$ leads to a polynomial $P_2$ whose behavior at $x = 0$ matches the behavior of $f$ as $x = 1$.

(4) Finally, we shift $P_2$ to the right by 1 in order to have the behavior of $P_2$ at $x = 1$ match the behavior of $f$ at $x = 1$. Explain why the resulting polynomial is $P_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$.

(5) Use technology (such as https://www.desmos.com/calculator) to make a plot showing both $f$ and $P_2$. Zoom in on a region centered at $x = 1$. How good is this approximation?

(6) Find a quadratic approximation of the function $g(x) = \frac{1}{x}$ near $x = 1$.

(7) (Challenge) What should we do if we want to approximate a function using a cubic polynomial? What about a quartic?