ASSIGNMENT 8

Linear approximations

Reading: The first two pages of section 3.1 in Smith–Minton. (Technically this stuff is also in section 2.1, but that notation in that section is weird.)

Approximation problem:

- Given a function $f(t)$, approximate f by an "easy" function near input t_*.
- Idea: approximate by a linear function $P_1(t)$

Linear approximation

- We can compute $f(t_*)$ and $f'(t_*)$.
- Start with a function having same rate of change as f does at time t_*.
- Adjust so that it is centered at the point $(t_*, f(t_*))$.
- Result is *first order approximation (centered at t_*)*

$$P_1(t) = f(t_*) + f'(t_*)\,(t - t_*)$$

- The book refers to these are "tangent lines"

Examples

- Find the first order approximation of $f(t) = \dfrac{1}{\sqrt{t}}$ centered at $t = 9$
- Find the first order approximation of $f(t) = \dfrac{5 - t^2}{\sqrt{t}}$ centered at $t = 4$
- Find the first order approximation of $f(x) = (x + 7)^2$ at $x = 3$

Revisit previous example

- Note that $f(3 + h) = 100 + 20h + h^2 = f(3) + f'(3)\,h + \dots$
- For a general function we have

$$f(x_* + h) = f(x_*) + f'(x_*)\,h + \dots$$

- We can reverse the logic. Compute

$$f(x_* + h) = f(x_*) + \boxed{}\,h + \text{stuff that vanishes}\dots$$

Whatever appears in the box is $f'(x_*)$.

Exercise 8.1. Find the first order approximation of the function $f(x) = \sqrt{x}$ centered at $x = 1$

Solution.

The first order approximation centered at x_* is $P_1(x) = f(x_*)+f'(x_*)(x-x_*)$. Here $x_* = 1$.

We compute $f(1) = 1$ and $f'(1) = \frac{1}{2}$. Thus the approximation is $P_1(x) = 1 + \frac{1}{2}(x-1)$.

Exercise 8.2. Find the first order approximation of the function $g(x) = \frac{1}{x}$ centered at $x = 1$.

Solution.

We compute $g(1) = 1$ and $g'(1) = -1$. Thus $P_1(x) = 1 - (x-1)$.

Exercise 8.3. Find the first order approximation of the function $f(t) = \frac{t^2-7\sqrt{t}}{t^3}$ centered at $t = 4$.

Solution.

Writing $f(t) = t^{-1} - 7t^{-5/2}$ we compute $f'(t) = \frac{35}{2t^{7/2}} - \frac{1}{t^2}$.
Thus $f(4) = \frac{1}{32}$ and $f'(4) = \frac{19}{256}$.
Hence $P_1(t) = \frac{1}{32} + \frac{19}{256}(t-4)$.

Exercise 8.4. In this exercise we explore the relationship between first order approximations and multiplication.

(1) Find the first order approximation of the function $h(x) = \frac{1}{\sqrt{x}}$ centered at $x = 1$.

(2) Explain why $h(x) = f(x)\,g(x)$, where f and g are from Exercises 8.1 and 8.2.

(3) Suppose you multiply together the approximations you found in Exercises 8.1 and 8.2. How does this compare to what you found in the part (1)?

Solution.

(1) We have $h(1) = 1$ and $h'(1) = -\frac{1}{2}$. Thus $P_1(x) = 1 - \frac{1}{2}(x-1)$.

(2) We simply compute $f(x)g(x) = \sqrt{x}\frac{1}{x} = \frac{1}{\sqrt{x}}$.

(3) Notice that

$$\left(1 + \frac{1}{2}(x-1)\right)\left(1 - (x-1)\right) = \underbrace{1 - \frac{1}{2}(x-1)} - \underbrace{\frac{1}{2}(x-1)^2}$$

where the first underbraced stuff is the first order approximation of h and the other stuff is quadratic.