

# Introduction to Mathematica

## Part II: Functions and Algebra

### Functions

We can define mathematical functions like  $f(x) = 2x$  and  $g(x) = \sin(3x)$  easily in *Mathematica*. This frees us from constantly typing the same expression, and makes substituting in values very easy. In this section, we'll learn several ways to define functions and then return to plotting to show how they can be used. Here's a simple example to start with:

```
f[x_] := 2x + 1;  
f[3]
```

The output will be

```
7
```

This defines a new function called `f` which works like the mathematical function  $f(x) = 2x + 1$ . As we can see, evaluating `f[3]` gives 7 as we expect. There are some important things to notice about how we define functions. First, the arguments on the left hand side must be followed by underscores (“\_”). Here, this indicates that “ $x$ ” should be treated as a placeholder variable. Also, the colon before the equals sign is required to indicate that we are defining a function.

**Exercise** Define the function  $g(x) = \sin(3x)$  and evaluate it at  $x = \frac{\pi}{6}$ .

We can also define functions that take two or more arguments in a similar way. Here's an example:

```
h[x_, y_] := Sqrt[x^2 + y^2];  
h[3, 4]
```

**Exercise** Define the function  $b(u, v) = \sin(u) \cos(v)$  and evaluate it at  $u = \frac{\pi}{3}$ ,  $v = \frac{\pi}{4}$ .

Another common task in calculus and other areas of math is to use piecewise functions, where different formulas are used depending on the arguments. For example, we can define a function  $w(x)$  that is equal to  $x^2$  when  $0 \leq x \leq 2$  and is equal to  $2x$  when  $x > 2$ . Here's how to enter such a function in *Mathematica*:

```
w[x_] := Piecewise[{{x^2, 0 ? x ? 2}, {2x, x > 2}}];  
{w[3/2], w[3]}
```

The output will be

```
{ $\frac{9}{4}$ , 6}
```

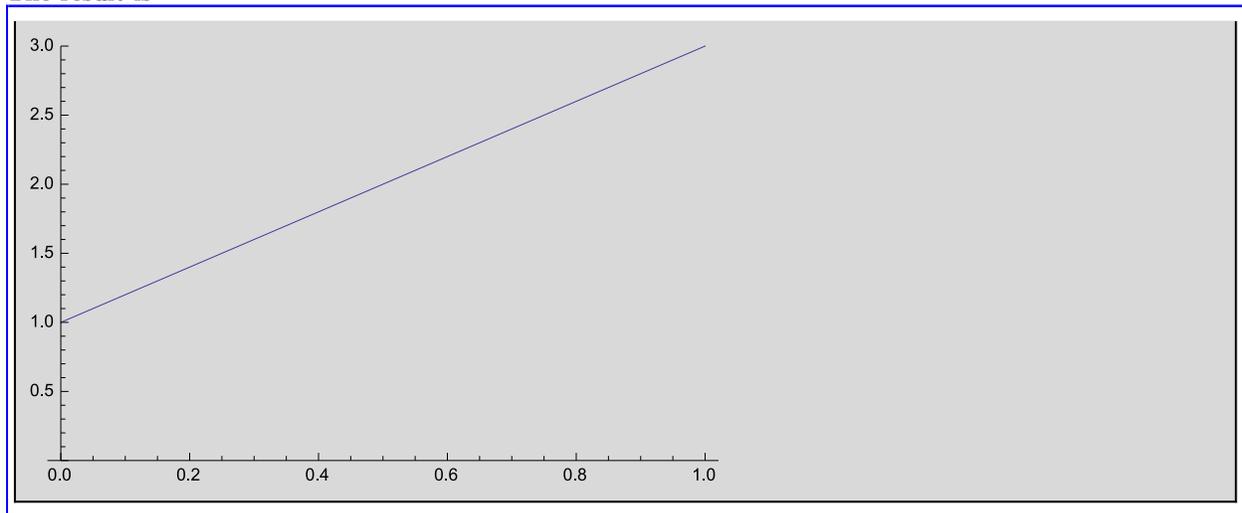
This uses the built-in function `Piecewise`, which takes a list of pairs `{output, condition}` and returns the output for the condition that is true. So, in our example, when we evaluate `w[3/2]`, the condition  $0 \leq 3/2 \leq 2$  is true so `Piecewise` returns  $(3/2)^2$  which is equal to  $9/4$ .

**Exercise** Define your own function  $q(x)$  which is equal to  $x^3 - 1$  when  $x \leq 0$  and is equal to  $x - 1$  when  $x > 0$ . Evaluate  $q(-2)$ .

Now that we've defined some functions, let's plot them. This is just like plotting a built-in function like `Sin` or `Sqrt`.

```
Plot[f[x], {x,0,1}, PlotRange->{0, 3}]
```

The result is

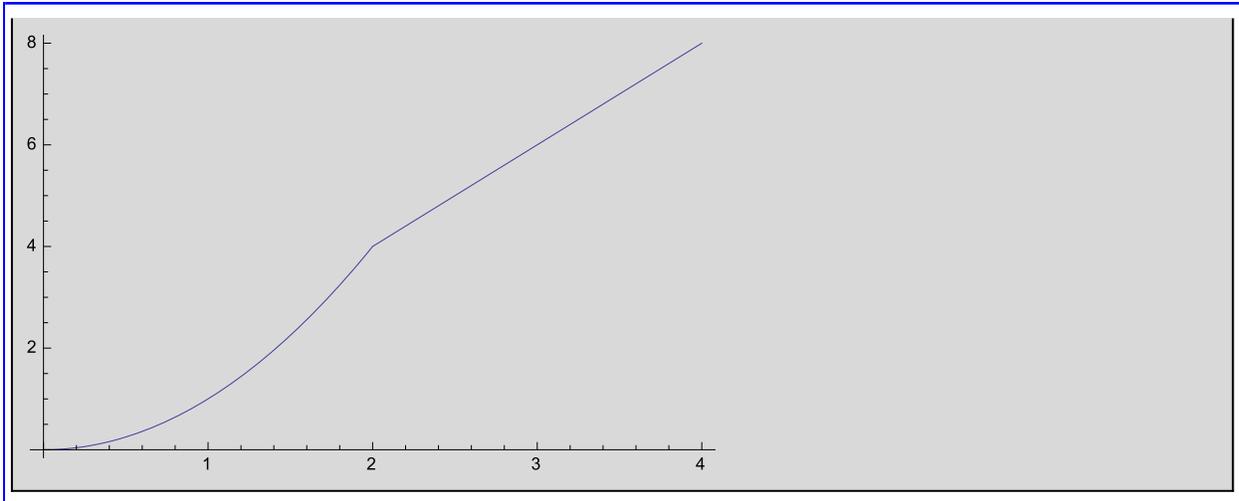


**Exercise** Plot the function  $g(x)$  that you defined above from  $x = 0$  to  $x = 2\pi$ .

We can also plot piecewise functions in the exact same way:

```
Plot[w[x], {x,0,4}]
```

The result is



**Exercise** Plot the function  $q(x)$  that you defined above on the range  $x = -1$  to  $x = 1$ .

## Algebra and Symbolic Manipulation

You might have noticed that in the previous part when we graphed functions we gave them arguments that weren't actually numbers. Instead, we wrote things like  $w[x]$  where  $x$  is what is called a symbol. This is like a variable in normal mathematics: something which stands for an undetermined value. *Mathematica* is very good at manipulating such symbolic expressions. Suppose we wanted to make the substitution  $x = \frac{1}{2}u$  in our function  $f(x)$  that we defined above. *Mathematica* can do this and simplify the answer for us:

```
Simplify[f[1/2u]]
```

The output is

```
1 + u
```

As we expected we get the answer  $1 + u$  because  $f\left(\frac{1}{2}u\right) = 2\left(\frac{1}{2}u\right) + 1 = u + 1$ . The important thing to notice is that we never gave  $u$  a value in this computation. We used the function `Simplify` here to make sure that *Mathematica* simplified the resulting expression as much as it could.

**Exercise** Make the substitution  $x = u^2$  in the function  $q(x)$  that you defined above. Be sure to simplify. Is this the result you were expecting?

Another common algebraic task that *Mathematica* excels at is solving equations. Here's an example:

```
Solve[f[x] == 0, x]
```

The result is

```
( x -> -1/2 )
```

This finds the value of  $x$  such that  $f(x) = 0$ . As we expect, there is one answer  $x = -\frac{1}{2}$ . The function `Solve` that we used here takes two arguments. The first is an equation, or a list of equations, to solve. The second is a variable, or a list of variables, that will be solved for.

We didn't necessarily have to use a function in the arguments to Solve. Instead, we can write out an equation directly.

```
quadraticSolutions=Solve[x^2-5x+6==0,x]
```

The result is

```
{x -> 2, x -> 3}
```

Because this equation was a quadratic, it gave us two solutions:  $x = 3$  and  $x = 2$ .

**Exercise** Find the roots of the polynomial  $-12 + 20x - 7x^2 - 2x^3 + x^4$ .

You might be wondering what we can do with the answers that Solve gives us. For instance, can I substitute the results into another equation or use them as arguments to a function? Yes, but we have to use what's called replacement. The answers that solve gives are in the form of lists of rules, which specify that a variable should become a given value. To apply such rules we use the `/.` command as follows:

```
x^2/.quadraticSolutions
```

The result of which is

```
{4,9}
```

What happened here? When I solve the quadratic equation above, I stored its answer in the variable `quadraticSolutions`. There are two rules solution, either  $x$  becomes 2 or  $x$  becomes 3. *Mathematica* went through consecutively and set the  $x$  in  $x^2$  to 2, giving 4, and then to 3, giving 9. The result was a list of possible values that  $x^2$  could become.

We can do the same thing with functions instead of directly entered expressions.

```
f[x]/.Solve[x^2-1==0,x]
```

The result is

```
{-1,3}
```

Here, there were two solutions for  $x$ : either  $x = -1$  or  $x = 1$ . These were then given to  $f$  as arguments, giving  $f(-1) = -1$  and  $f(1) = 3$ .

**Exercise** Use *Mathematica* to solve the following problem: An orange weighs  $\frac{2}{5}$  of a pound and costs \$0.76. If I buy 6 pounds of orange, how much will I spend?