DAY 20

Nonlinear ODE: The method of nullclines

Reading: §5.2 Qualitative Analysis

Goal: Study non-linear system away from equilibria.

- Nullclines: Those places where motion is either vertical or horizontal; these will be curves (or lines) in phase portrait.
- Horizontal motion $\leftrightarrow \frac{dy}{dt} = 0$
- Vertical motion $\leftrightarrow \frac{dx}{dt} = 0$
- These act as one-way doors in state space.

Example from yesterday: $\dfrac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy \qquad \dfrac{dy}{dt} = -y + xy$

- Horizontal motion when $0 = y(-1 + x)$. Happens on lines $x = 1$ and $y = 0$. Draw picture.
- Notice what happens at equilibria: no motion
- In between equilibria direction of motion does not change: use equation to determine direction.
- Along $y = 0$ we have $\frac{dx}{dt} = 2x(1 - \frac{x}{2})$. Positive when $0 < x < 2$; negative otherwise.
- Along $x = 1$ we have $\frac{dx}{dt} = 1 - y$. Positive when $y > 1$, negative otherwise.
- Vertical motion when $0 = 2x\left(1 - \frac{x}{2}\right) - xy$. Happens on lines $x = 0$, $y = -x + 2$
- Along $x = 0$ we have $\frac{dy}{dt} = -y$. Positive when $y < 0$; negative otherwise.
- Along $y = -x$ we have $\frac{dy}{dt} = \ldots$

Exercise 20.1. Extract as much information as possible about the systems of Day 19 Problem 19.1 by using the method of nullclines. Contrast and compare what you were able to conclude using linearization but you were not able to conclude using nullclines, and vice versa. Then write a one-paragraph conclusion about the benefit of using both methods: one for understanding the local behavior and one for understanding the global behavior.

Solution:

(1) **Horizontal motion nullclines:**
These are curves where $\frac{dy}{dt} = 0$. This happens when

$$0 = -2y + 2xy = -2y(1 - x).$$

Thus we have two nullclines, $y = 0$ and $x = 1$.
Along $y = 0$**::** Here we have

$$\frac{dx}{dt} = x \text{ which is } \begin{cases} > 0 & \text{when } x > 0 \\ < 0 & \text{when } x < 0 \end{cases}$$

Along $x = 1$**::** Here we have $\frac{dx}{dt} = x > 0$.
Vertical motion nullclines:
These are curves where $\frac{dx}{dt} = 0$. This happens when $x(1 - y) = 0$.
Thus we consider two lines: $x = 0$ and $y = 1$.
Along $y = 1$**::** We have

$$\frac{dy}{dt} = -2 + 2x \text{ which is } \begin{cases} > 0 & \text{when } x > 1 \\ < 0 & \text{when } x < 1 \end{cases}$$

Along $x = 0$**::** We have

$$\frac{dy}{dt} = -2y \text{ which is } \begin{cases} > 0 & \text{when } y < 0 \\ < 0 & \text{when } y > 0 \end{cases}$$

(2) (no solution provided. . . yet)

(3) **Horizontal motion nullclines:**
These are curves where $\frac{dy}{dt} = 0$. This happens when along the lines

$$y = 0$$

and

$$\frac{1}{2} + \frac{1}{100}x - \frac{1}{400}y = 0.$$

Along $y = 0$**::** Here we have

$$\frac{dx}{dt} = x\left(2 - \frac{1}{50}x\right) \text{ which is } \begin{cases} > 0 & \text{when } 0 < x < 100 \\ < 0 & \text{when } x < 0 \text{ or } x > 100 \end{cases}$$

Along $\frac{1}{2} + \frac{1}{100}x - \frac{1}{400}y = 0$**::** Here we have

$$\frac{dx}{dt} = x\left(1 - \frac{1}{25}x\right) \text{ which is } \begin{cases} > 0 & \text{when } 0 < x < 25 \\ < 0 & \text{when } x < 0 \text{ or } x > 25 \end{cases}$$

<u>Vertical motion nullclines:</u>

These are curves where $\frac{dx}{dt} = 0$. This happens along the lines $x = 0$ and $2 - \frac{1}{50}x - \frac{1}{200}y = 0$.

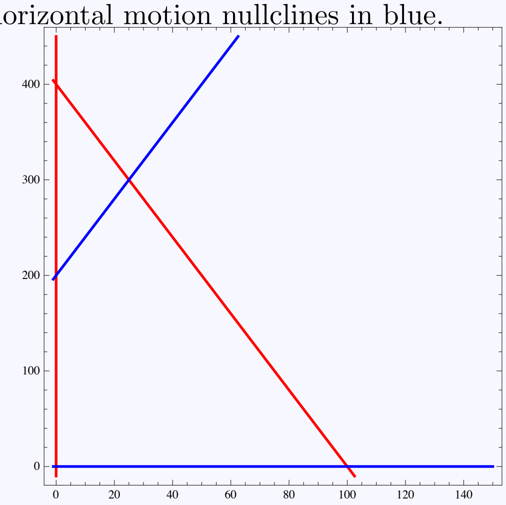
Along $x = 0$**::** We have

$$\frac{dy}{dt} = \frac{1}{2}y\left(1 - \frac{y}{200}\right) \text{ which is } \begin{cases} > 0 & \text{when } 0 < y < 200 \\ < 0 & \text{when } y < 0 \text{ or } y > 200 \end{cases}$$

Along $2 - \frac{1}{50}x - \frac{1}{200}y = 0$**::** We have

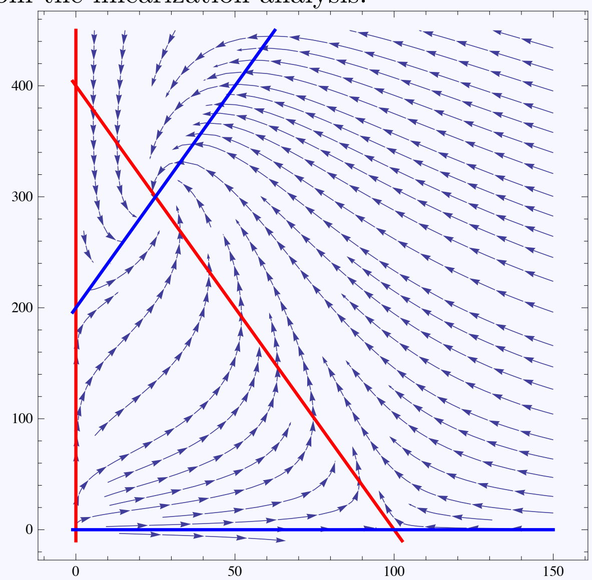
$$\frac{dy}{dt} = \frac{1}{2}y\left(3 - \frac{y}{100}\right) \text{ which is } \begin{cases} > 0 & \text{when } 0 < y < 300 \\ < 0 & \text{when } y > 300 \text{ or } y < 0 \end{cases}$$

Here is a picture of the nullclines, with vertical motion nullclines in red and horizontal motion nullclines in blue.



Notice that there are two zones where we can exactly control the entry/exit of solution curves. This nicely complements the picture we obtain from linearization.

When the nullcline plot is superimposed on the StreamPlot we have the picture below, which is consistent with the picture we obtain from the linearization analysis.



Exercise 20.2. The following system of equations models the populations (in millions) of two competing animal species:

$$\begin{cases} \dfrac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy \\ \\ \dfrac{dy}{dt} = 4y\left(1 - \frac{y}{4}\right) - 3xy. \end{cases}$$

(1) Find all equilibrium solutions of this system.
(2) Analyze the system near the equilibrium solutions by means of linearization.
(3) Analyze the non-linear system using nullclines.
(4) Using all this work complete (as much as possible) the phase portrait of the full non-linear system.