DAY 19

Non-linear ODE: Linearization & equilibrium point analysis

Reading: §5.1 Equilibrium Point Analysis

Plan for analyzing non-linear system

- Identify equilibrium points
- Linearize about each point, obtaining a local portrait
- Assemble in to a global picture

Example: $\frac{dx}{dt} = \underbrace{2x\left(1 - \frac{x}{2}\right) - xy}_{f(x,y)} \qquad \frac{dy}{dt} = \underbrace{-y + xy}_{g(x,y)}$

- Equilibria at $(0,0)$, $(2,0)$, $(1,1)$
- At $(1,1)$: Set $w = x - 1$, $v = x - 1$ and obtain linearization

$$\frac{d}{dt}\begin{pmatrix} w \\ v \end{pmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} w \\ v \end{pmatrix}$$

  Counter-clockwise spiral sink with $\lambda = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

- At $(0,0)$: Set $w = x - 0$, $v = x - 0$ and obtain linearization

$$\frac{d}{dt}\begin{pmatrix} w \\ v \end{pmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} w \\ v \end{pmatrix}$$

  Saddle with $\lambda = 2$ having $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda = -1$ having $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- At $(2,0)$: Set $w = x - 2$, $v = x - 0$ and obtain linearization

$$\frac{d}{dt}\begin{pmatrix} w \\ v \end{pmatrix} = \begin{bmatrix} -2 & 02 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} w \\ v \end{pmatrix}$$

  Saddle with $\lambda = -2$ having $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda = 1$ having $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

- Assemble global picture. Interpret as predator-prey. . .

Exercise 19.1. Perform the equilibrium point analysis of the following predator-prey models. That is, execute the following steps.

- Find all equilibrium solutions of the system.
- Linearize the system near each equilibrium.
- Understand the linearized models using the "eigen-techniques" you learned earlier.
- As much as possible, piece the phase portraits of the linearized systems together to get an approximate phase portrait of the full non-linear system. Confirm with Mathematica.

(1) $$\begin{cases} \dfrac{dx}{dt} = x - xy \\ \dfrac{dy}{dt} = -2y + 2xy \end{cases}$$

(2) $$\begin{cases} \dfrac{dP}{dt} &= 0.3P\big(1 - \dfrac{P}{100}\big) - 0.06PR \\ \dfrac{dR}{dt} &= -0.4R + 0.01PR. \end{cases}$$

(3) $$\begin{cases} \dfrac{dx}{dt} = 2x\left(1 - \dfrac{x}{100}\right) - 0.005xy \\ \dfrac{dy}{dt} = \dfrac{y}{2}\left(1 - \dfrac{y}{200}\right) + 0.01xy. \end{cases}$$

Solution:

(1) Any equilibrium solutions $x(t) \equiv C, y(t) \equiv D$ arise from the system of algebraic equations

$$\begin{cases} 0 = C - CD = C(1-D) \\ 0 = -2D + 2CD = -2D(1-C). \end{cases}$$

The first equation is fulfilled in one of the two of the following circumstances.

Circumstance 1:: If $C = 0$ then the first equation automatically holds. Inserting this information into the second equation produces $0 = -2D$, i.e. $D = 0$. So, the equilibrium solution in this case is

$$x(t) \equiv 0, \ y(t) \equiv 0.$$

Circumstance 2:: If $1 - D = 0$ then the first equation also holds. From here we get $D = 1$ which, when inserted into the second equation, yields $C = 1$. The second equilibrium is

$$x(t) \equiv 1, \; y(t) \equiv 1.$$

To proceed with the linearization we introduce some notation. We let $\mathcal{F}(x, y) = x - xy$ and let $\mathcal{G}(x, y) = -2y + 2xy$. We have

$$\frac{d\mathcal{F}}{dx} = 1 - y, \qquad \frac{d\mathcal{F}}{dy} = -x$$
$$\frac{d\mathcal{G}}{dx} = 2y, \qquad \frac{d\mathcal{G}}{dy} = -2 + 2x.$$

Linearization near $(0, 0)$*:* Since $\mathcal{F}(0, 0) = 0$, $\frac{d\mathcal{F}}{dx}(0, 0) = 1$ and $\frac{d\mathcal{F}}{dy}(0, 0) = 0$, we have

$$\mathcal{F}(0 + h, 0 + k) \approx h.$$

Since $\mathcal{G}(0, 0) = 0$, $\frac{d\mathcal{G}}{dx}(0, 0) = 0$ and $\frac{d\mathcal{G}}{dy}(0, 0) = -2$, we have

$$\mathcal{G}(0 + h, 0 + k) \approx -2k.$$

Thus, our linearized system is

$$\begin{cases} \frac{dh}{dt} = h \\ \\ \frac{dk}{dt} = -2k. \end{cases}$$

The matrix of this system is

$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

Its eigenvalues are easily found to be $\lambda_1 = 1$ and $\lambda_2 = -2$. An eigenvector $\vec{V}_1 = \begin{pmatrix} v \\ w \end{pmatrix}$ for the eigenvalue of $\lambda_1 = 1$ satisfies $-2w = 0$, i.e. $w = 0$. Thus, we may choose

$$\vec{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

An eigenvector $\vec{V}_2 = \begin{pmatrix} v \\ w \end{pmatrix}$ for the eigenvalue of $\lambda_2 = -2$ satisfies $3v = 0$, i.e. $v = 0$. Thus, we may choose

$$\vec{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, the phase diagram here is a saddle whose incoming axis is the vertical $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and whose out going axis is the horizontal $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Linearization near $(1, 1)$*:* Since $\mathcal{F}(1, 1) = 0$, $\frac{d\mathcal{F}}{dx}(1, 1) = 0$ and $\frac{d\mathcal{F}}{dy}(1, 1) = -1$, we have

$$\mathcal{F}(1 + h, 1 + k) \approx -k.$$

Since $\mathcal{G}(1, 1) = 0$, $\frac{d\mathcal{G}}{dx}(1, 1) = 2$ and $\frac{d\mathcal{G}}{dy}(1, 1) = 0$, we have

$$\mathcal{G}(1 + h, 1 + k) \approx 2h.$$

Using the substitutions $x(t) = 1 + h(t)$, $y(t) = 1 + k(t)$ we see that our linearized system is

$$\begin{cases} \frac{dh}{dt} = -k \\ \\ \frac{dk}{dt} = 2h. \end{cases}$$

The matrix of this system is

$$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$$

We see from

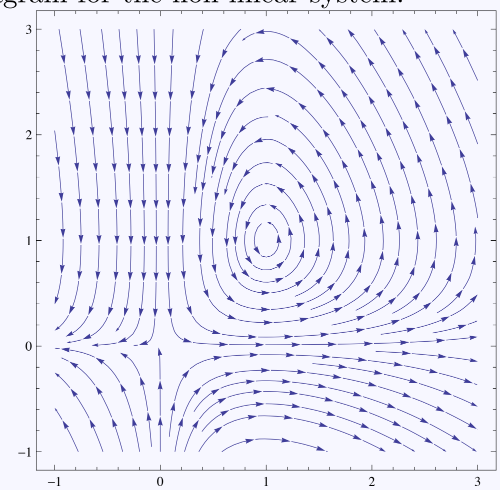
$$(-\lambda)(-\lambda) + 2 = 0$$

that its eigenvalues are $\lambda = \pm\sqrt{2}$. Thus, the phase diagram here is a cycles of sorts (circles, tilted ellipses etc.). To find if these cycles are traversed in the clock-wise or counter-clockwise direction we observe that at the point $(h, k) = (1, 0)$ the phase curves go in the direction of

$$\begin{pmatrix} \frac{dh}{dt} \\ \\ \frac{dk}{dt} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

i.e. vertically up. Thus, the cycles are counter-clockwise.

Putting pieces together: Combining the saddle at the equilibrium point $(0,0)$ with the counter-clockwise cycle at $(1,1)$ we get the following diagram for the non-linear system:



(2) Any equilibrium solutions $P(t) \equiv C, R(t) \equiv D$ arise from the system of algebraic equations

$$\begin{cases} 0 = 0.3C(1 - 0.01C) - 0.06CD = 0.3C(1 - 0.01C - 0.2D) \\ 0 = -0.4D + 0.01CD = 0.1D(-4 + 0.1C). \end{cases}$$

The first equation is fulfilled in one of the two of the following circumstances.

Circumstance 1: If $C = 0$ then the first equation automatically holds. Inserting this information into the second equation produces $0 = -0.4D$, i.e. $D = 0$. So, the equilibrium solution in this case is

$$P(t) \equiv 0, \ R(t) \equiv 0.$$

Circumstance 2: If $1 - 0.01C - 0.2D = 0$ then the first equation also holds. From here we get $C = 100 - 20D$ which, when inserted into the second equation, yields

$$0 = 0.1D(6 - 2D).$$

The last equation has solutions of $D = 0$ (in which case $C = 100$) and $D = 3$ (in which case $C = 40$). So, the equilibrium solutions in this case are

$$P(t) \equiv 100, \ R(t) \equiv 0 \quad \text{and} \quad P(t) \equiv 40, \ R(t) \equiv 3.$$

To proceed with the linearization we introduce some notation. We let $\mathcal{F}(P,R) = 0.3P\big(1 - \frac{P}{100}\big) - 0.06PR$ and let $\mathcal{G}(P,R) = -0.4R + 0.01PR$. We have

$$\frac{d\mathcal{F}}{dP} = 0.3 - 0.006P - 0.06R, \qquad \frac{d\mathcal{F}}{dR} = -0.06P$$
$$\frac{d\mathcal{G}}{dP} = 0.01R, \qquad \frac{d\mathcal{G}}{dR} = -0.4 + 0.01P.$$

Linearization near $(0,0)$*:* Since $\mathcal{F}(0,0) = 0$, $\frac{d\mathcal{F}}{dP}(0,0) = 0.3$ and $\frac{d\mathcal{F}}{dR}(0,0) = 0$, we have

$$\mathcal{F}(0+h, 0+k) \approx 0.3h.$$

Since $\mathcal{G}(0,0) = 0$, $\frac{d\mathcal{G}}{dP}(0,0) = 0$ and $\frac{d\mathcal{G}}{dR}(0,0) = -0.4$, we have

$$\mathcal{G}(0+h, 0+k) \approx -0.4k.$$

Thus, our linearized system is

$$\begin{cases} \frac{dh}{dt} = 0.3h \\ \\ \frac{dk}{dt} = -0.4k. \end{cases}$$

The matrix of this system is

$$\begin{pmatrix} 0.3 & 0 \\ 0 & -0.4 \end{pmatrix}$$

Its eigenvalues are easily found to be $\lambda_1 = 0.3$ and $\lambda_2 = -0.4$. An eigenvector $\vec{V}_1 = \begin{pmatrix} v \\ w \end{pmatrix}$ for the eigenvalue of $\lambda_1 = 0.3$ satisfies $-0.4w = 0$, i.e. $w = 0$. Thus, we may choose

$$\vec{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

An eigenvector $\vec{V}_2 = \begin{pmatrix} v \\ w \end{pmatrix}$ for the eigenvalue of $\lambda_2 = -0.4$ satisfies $0.7v = 0$, i.e. $v = 0$. Thus, we may choose

$$\vec{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, the phase diagram here is a saddle whose incoming axis is the vertical $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and whose out going axis is the horizontal $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Linearization near $(100, 0)$*:* Since $\mathcal{F}(100, 0) = 0$, $\frac{d\mathcal{F}}{dP}(100, 0) = -0.3$ and $\frac{d\mathcal{F}}{dR}(100, 0) = -6$, we have

$$\mathcal{F}(100 + h, 0 + k) \approx -0.3h - 6k.$$

Since $\mathcal{G}(100, 0) = 0$, $\frac{d\mathcal{G}}{dP}(100, 0) = 0$ and $\frac{d\mathcal{G}}{dy}(100, 0) = 0.6$, we have

$$\mathcal{G}(100 + h, 0 + k) \approx 0.6k.$$

Using the substitutions $x(t) = 100 + h(t)$, $y(t) = 0 + k(t)$ we see that our linearized system is

$$\begin{cases} \frac{dh}{dt} = -0.3h - 6k \\ \\ \frac{dk}{dt} = 0.6k. \end{cases}$$

The matrix of this system is

$$\begin{pmatrix} -0.3 & -6 \\ 0 & 0.6 \end{pmatrix}$$

We see from

$$(0.6 - \lambda)(-0.3 - \lambda) = 0$$

that its eigenvalues are $\lambda_1 = 0.6$ and $\lambda_2 = -0.3$. An eigenvector $\vec{V_1} = \begin{pmatrix} v \\ w \end{pmatrix}$ for the eigenvalue of $\lambda_1 = 0.6$ satisfies $-0.9v - 6w = 0$, i.e. $w = -0.15v$. Thus, we may choose

$$\vec{V_1} = \begin{pmatrix} 20 \\ -3 \end{pmatrix}.$$

An eigenvector $\vec{V_2} = \begin{pmatrix} v \\ w \end{pmatrix}$ for the eigenvalue of $\lambda_2 = -0.3$ satisfies $w = 0$. Thus, we may choose

$$\vec{V_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus, the phase diagram here is a saddle whose incoming axis is the horizontal $\begin{pmatrix}1\\0\end{pmatrix}$, and whose out going axis is spanned by the vector $\begin{pmatrix}20\\-3\end{pmatrix}$.

Linearization near $(40,3)$*:* Since $\mathcal{F}(40,3)=0$, $\frac{d\mathcal{F}}{dP}(40,3)=-0.12$ and $\frac{d\mathcal{F}}{dR}(40,3)=-2.4$, we have

$$\mathcal{F}(40+h,3+k)\approx -0.12h-2.4k.$$

Since $\mathcal{G}(40,3)=0$, $\frac{d\mathcal{G}}{dP}(40,3)=0.03$ and $\frac{d\mathcal{G}}{dR}(40,3)=0$, we have

$$\mathcal{G}(40+h,3+k)\approx 0.03h.$$

Using the substitutions $x(t)=40+h(t)$, $y(t)=3+k(t)$ we see that our linearized system is

$$\begin{cases}\frac{dh}{dt}=-0.12h-2.4k\\[2em]\frac{dk}{dt}=0.03h.\end{cases}$$

The matrix of this system is

$$\begin{pmatrix}-0.12 & -2.4\\ 0.03 & 0\end{pmatrix}$$

We see from

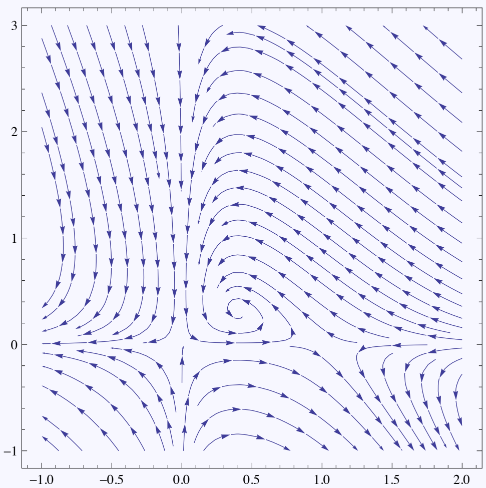
$$(-\lambda)(-0.12-\lambda)+0.072=0,\quad \text{i.e.}\quad \lambda^2+0.12\lambda+0.072=0$$

that its eigenvalues are $\lambda=-0.06\pm\sqrt{.0684}i$. Thus, the phase diagram here is a spiral sink! To find if these spirals are traversed in the clock-wise or counter-clockwise direction we observe that at the point $(h,k)=(1,0)$ the phase curves go in the direction of

$$\begin{pmatrix}\frac{dh}{dt}\\[2em]\frac{dk}{dt}\end{pmatrix}=\begin{pmatrix}-0.12\\0.03\end{pmatrix},$$

i.e. counter-clockwise.

Putting pieces together: Combining the saddle at the equilibrium point $(0,0)$, with the saddle at $(100,0)$, with the counter-clockwise spiral sink at $(40, 3)$ we get the following diagram for the non-linear system:



(3) It is straightforward to find the equilibria for this system:

- $(x, y) = (0, 0)$
- $(x, y) = (0, 200)$
- $(x, y) = (100, 0)$
- $9x, y) = (25, 300)$.

We now linearize the system near each equilibrium. In preparation for this, we write the system as

$$\frac{dx}{dt} = f(x, y) = 2x - \frac{1}{50}x^2 - \frac{1}{200}xy$$
$$\frac{dy}{dt} = g(x, y) = \frac{1}{2}y - \frac{1}{400}y^2 + \frac{1}{100}xy$$

and compute

$$\frac{\partial f}{\partial x} = 2 - \frac{1}{25}x - \frac{1}{200}y$$
$$\frac{\partial f}{\partial y} = -\frac{1}{200}x$$
$$\frac{\partial g}{\partial x} = \frac{1}{100}x$$
$$\frac{\partial g}{\partial y} = \frac{1}{2} - \frac{1}{200}y + \frac{1}{100}x$$

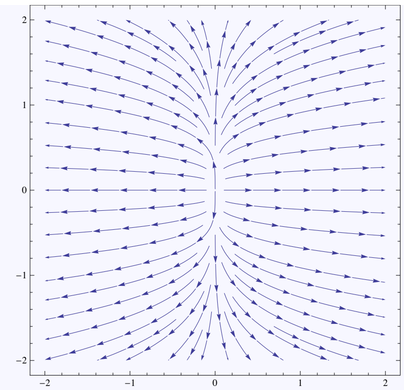
Linearization near $(x, y) = (0, 0)$:: Setting $u = x - 0$ and $v = y - 0$ we find the linearized system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

The eigenvalues are seen to be $\lambda = 2, 1/2$. Since they are positive, this is a source. We find the eigensolutions to be

$$e^2t\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e^{\frac{1}{2}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The phase plot for looks like



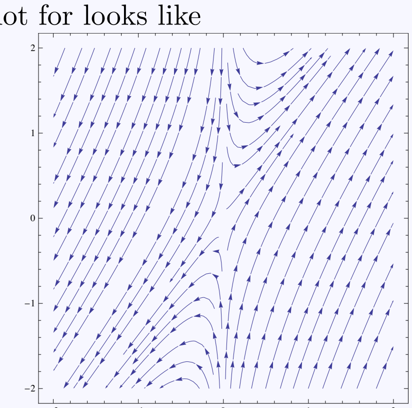
Linearization near $(x, y) = (0, 200)$**::** Setting $u = x - 0$ and $v = y - 200$ we find the linearized system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

The eigenvalues are $\lambda = 1$ and $\lambda = -1/2$; thus this equilibrium is a saddle. We find the eigensolutions to be

$$e^{t}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{and} \quad e^{-\frac{1}{2}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The phase plot for looks like



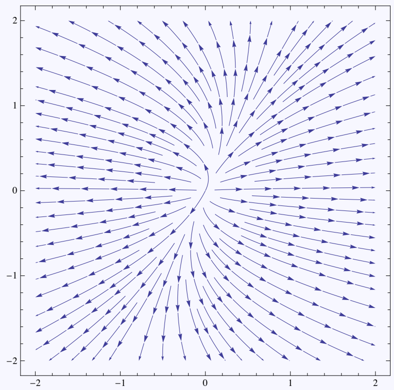
Linearization near $(x, y) = (100, 0)$**::** Setting $u = x - 100$ and $v = y - 0$ we find the linearized system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} -2 & -1/2 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

The eigenvalues are seen to be $\lambda = 2, 1$. Since they are positive, this is a source. We find the eigensolutions to be

$$e^{2t}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad e^{t}\begin{pmatrix}1\\2\end{pmatrix}.$$

The phase plot for looks like

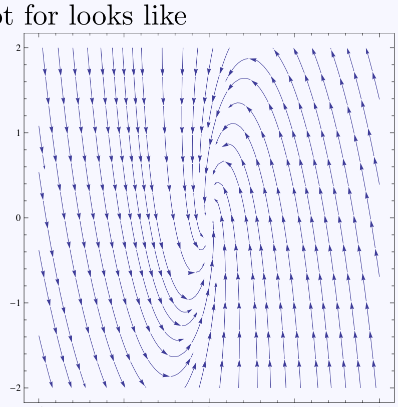


Linearization near $(x, y) = (25, 300)$**::** Setting $u = x - 25$ and $v = y - 30$ we find the linearized system is

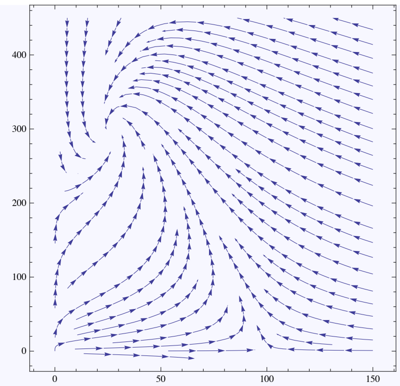
$$\frac{d}{dt}\begin{pmatrix}u\\v\end{pmatrix} = \begin{bmatrix}-1/2 & -1/8\\3 & -3/4\end{bmatrix}\begin{pmatrix}u\\v\end{pmatrix}$$

The eigenvalues are seen to be $\lambda = -\frac{5}{8} \pm \frac{\sqrt{23}}{8}\,i$. Since they are complex, with negative real part, this is a spiral sink. Examining the equation at $(u, v) = (1, 0)$ we see that the direction of rotation is counter-clockwise.

The phase plot for looks like



Finally we assemble these pieces together. The result is an approximation of the following phase diagram:



Exercise 19.2. Consider the non-linear system

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -x + (1 - x^2 - y^2)y.$$

(1) There is one equilibrium solution of this system – find it!

(2) Linearize the system near this equilibrium, and draw the phase portrait of the linearized system.

(3) Make an educated guess about the phase portrait of the non-linear system. *For your own benefit do this without any help of "technology".*

(4) Show that $x(t) = \sin(t)$, $y(t) = \cos(t)$ is a solution of the non-linear system. What is the phase diagram of this solution?

(5) Now make an educated guess about the phase portrait of the non-linear system. (Remember: phase curves for nice systems do not intersect!!!)

(6) Construct a phrase plot using Mathematica.

(7) Comment on what you learned about linearization from this problem.