3.2 The wave equation in one dimension

Exercise 3.2.1. Suppose that for $k = 1, \dots, N$ we have $\phi_k \colon [-1, 1] \to \mathbb{R}$ is a collection of orthogonal functions. Show that for any function $u \colon [-1, 1] \to \mathbb{R}$, the combination of the ϕ_k which best approximates u is given by

$$u_N = \sum_{k=1}^{N} \alpha_k \phi_k, \qquad \text{where} \qquad \alpha_k = \frac{\langle \phi_k, u \rangle}{\|\phi_k\|^2}.$$

Exercise 3.2.2. We now want to apply the previous exercise to the case where the ϕ_k the Dirichlet cosines and sines. This is a bit tricky because we have to counters. Recall that

$$\begin{aligned} v_k &= \cos\left(\frac{2k+1}{2}\pi x\right), & k &= 0, 1, 2 \dots, \\ w_k &= \sin(k\pi x), & k &= 1, 2, 3, \dots. \end{aligned}$$

Thus we seek to approximate a function u by

$$u_N = \sum_{k=0}^{N} \alpha_k v_k + \sum_{k=1}^{N} \beta_k w_k.$$

1. What formulas for α_k and β_k make u_N the best approximation of u?
2. Suppose $u(x) = 1 - |x|$. Compute the coefficients α_k and β_k. Make plots of u_2, u_4, u_{10}, and u_{20}, and compare those plots to the plot of u.
3. Repeat for the function

$$u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Exercise 3.2.3.

1. What is the Dirichlet initial boundary value problem (IBVP) for the wave equation on the domain $\Omega = [-1, 1]$?
2. Solve the IBVP with $u_0(x) = \sin(3\pi x)$, $v_0(x) = \cos(\frac{1}{2}\pi x)$.