Chapter 3

The wave equation

3.1 Preliminaries

TNDG

Exercise 3.1.1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x, y) = -4x^2 + 2xy + \frac{5}{2}y^2.$$

In case you're wondering, here's how I built this function:

$$\frac{1}{2}f(x, y) = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -8 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In any event, find the "Euler-Lagrange equation" for this function, but finding the gradient, and then setting it to zero.

Exercise 3.1.2.

1. Explain how to find solutions to differential equations of the form $\frac{d}{dt}Y = MY$, where Y is a vector in $\mathbb{R}^2$ and M is a matrix of constants.
2. Consider the equation

$$\frac{d}{dt}Y = \begin{pmatrix} -8 & 2 \\ 2 & 5 \end{pmatrix} Y.$$

- Find the general solution to the equation.
- Solve the initial value problem with initial condition

$$Y(0) = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.$$

3. Explain why the differential equation takes the form

$$\frac{d}{dt}Y = \operatorname{grad} f(Y),$$

where f is the function from Exercise 3.1.1.

Exercise 3.1.3. Now we explore equations of the form

$$\frac{d^2}{dt^2}Y = MY.$$

1. Consider the equation

$$\frac{d^2}{dt^2}Y = \begin{pmatrix} -8 & 2 \\ 2 & 5 \end{pmatrix} Y. \tag{3.1}$$

TWE

- Find all eigen-solutions to the equation.
- Find the general solution to the equation.
- Solve the initial value problem with initial conditions

$$Y(0) = \begin{pmatrix} -3 \\ 4 \end{pmatrix}. \quad \text{and} \quad Y'(0) = \begin{pmatrix} 16 \\ -17 \end{pmatrix}.$$

2. Show that the equation (3.1) is the Euler-Lagrange equation of the functional

$$A[Y] = \int_0^T \left(\frac{1}{2} \left\| \frac{d}{dt}Y \right\|^2 + f(Y) \right) dt,$$

where f is the function from Exercise 3.1.1.

Exercise 3.1.4. For functions $u = u(t, \mathbf{x})$ defined for t in $[0, T]$ and $\mathbf{x}$ in domain Ω, consider the action

$$A[u] = \int_0^T \int_\Omega \left\{ \frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 - \| \operatorname{grad} u \|^2 \right\} dt \, dV.$$

Show that the Euler-Lagrange equation is

$$\frac{\partial^2 u}{\partial t^2} = \Delta u,$$

which is called the *wave equation*.

Exercise 3.1.5. Show that the wave equation takes the following forms:

1. One dimension:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

2. Two dimensions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

3. Three dimensions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2 \phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cos \phi}{r^2 \sin \phi}\frac{\partial u}{\partial \phi}$$

Exercise 3.1.6. Here we explore the “standing wave solutions” to the wave equation. Here $\mathbf{x}$ represents all the spatial coordinates, in whatever coordinate system we happen to be using. We also assume that we have “acceptable” boundary conditions, which include:

- Dirichlet

- Neumann
- Periodic, in 1D setting

1. Show that $u = A(t)\psi(\mathbf{x})$ is a solution to the wave equation if
   - ψ is an eigenfunction of the Laplacian with eigenvalue λ, and
   - A satisfies $\frac{d^2A}{dt^2} = \lambda A$.
2. Use integration by parts to show that if ψ is an eigenfunction of the Laplacian with eigenvalue λ, then $\lambda \leq 0$ and hence we can write $\lambda = -\omega^2$.

   What should the definition of "acceptable" boundary conditions be?
3. Find the solutions to $\frac{d^2A}{dt^2} = -\omega^2 A$.
4. Suppose ψ is an eigenfunction of the Laplacian with eigenvalue $\lambda = -\omega^2$. What do the corresponding solutions to the wave equation look like? Why might they be called "standing wave" solution?

Exercise 3.1.7. Suppose we are working with one of the "acceptable" boundary conditions above, and that we have two different eigenfunctions ψ_1 and ψ_2 of the Laplacian. Show that if the corresponding eigenvalues are different, then ψ_1 and ψ_2 are orthogonal.

Exercise 3.1.8. Focus now on the one-dimensional situation.

1. We say that L is a linear one-dimensional second-order differential operator if

   $$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \tag{3.2}$$

   Generic1D

   for some functions a, b, c. Explain why the one-dimensional Laplacian is a second-order linear differential operator.
2. A differential operator L is called "symmetric" (or "self-adjoint") if

   $$\langle Lu, v\rangle = \langle u, Lv\rangle$$

   for any functions u and v.

Suppose we are working with the one-dimensional Laplacian on the domain $[-1, 1]$ with one of our three acceptable boundary conditions (Dirichlet, Neumann, Periodic). Show that it is symmetric.

3. Suppose we work on the domain $[-1, 1]$, but do not enforce any boundary conditions. Let

$$L = (1 - x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}. \tag{3.3}$$

LegendaryOperator

Show that L is symmetric.

(Why don't we need boundary conditions? Does this change your definition of "acceptable" boundary conditions?)

4. Show that if the operator in (3.2) is symmetric, then the function b is the derivative of the function a.

5. Suppose we have a symmetric operator of the form (3.2) with $c = 0$ and $a \geq 0$. Suppose also that we are working with "acceptable" boundary conditions for the operator. Show that all eigenvalues of the operator will be negative.

6. Show that $f(x) = 35x^4 - 30x^2 + 3$ is an eigenfunction for the operator (3.3). What is the corresponding eigenvalue?

   Repeat for $g(x) = 231x^6 - 315x^4 + 105x^2 - 5$.

   Then show that f and g are orthogonal.

7. Show in general that if we have a symmetric operator with acceptable boundary conditions then eigenfunctions with different eigenvalues will be orthogonal.