Exercise 2.3.1. A collection of objects is called a *(real) vector space* if

- (adding) there is a way of adding the object together, and
- (scaling) there is a way of multiplying the objects by real numbers

in such a way that all the "usual" algebraic properties hold. (This means that the distributive property holds, etc.)

1. Explain why $\mathbb{R}^n$ is a vector space in this sense.
2. Explain why the collection of polynomials is a vector space.
3. Let X_D be the collection of functions $u\colon [-1,1] \to \mathbb{R}$ such that $u(1) = 0$, $u(-1) = 0$. Explain why X_D is a vector space.

   We say that these functions satisfy "Dirichlet" boundary conditions.
4. Let X_N be the collection of functions $u\colon [-1,1] \to \mathbb{R}$ such that $u'(1) = 0$, $u'(-1) = 0$. Explain why X_N is a vector space.

   We say that these functions satisfy "Neumann" boundary conditions.
5. Let X_P be the collection of functions $u\colon [-1,1] \to \mathbb{R}$ such that $u(1) = u(-1)$ and $u'(1) = u'(-1)$. Explain why X_P is a vector space.

   We say that these functions satisfy "periodic" boundary conditions.
6. Look up "vector space" on Wikipedia. You'll find a long list of properties which vector spaces are supposed to have. I claim that the two bullet points above do a good job summarizing these properties. Do you agree? Explain.

Exercise 2.3.2. Let's explore these inner-products some more, focusing on the vector space X_D from the previous example.

1. Explain by

$$\begin{aligned} u_1(x) &= \sin(\pi x) \\ u_2(x) &= \sin(2\pi x) \\ &\vdots \\ u_k(x) &= \sin(k\pi x) \end{aligned}$$

are all in the space X_D.

Find an analogous list w_k of cosine functions which are in X_D.

2. Compute $\langle u_1, u_1\rangle$, $\langle u_1, u_2\rangle$, and $\langle u_2, u_2\rangle$. Then find a formula for $\langle u_k, u_l\rangle$. [Hint: Trigonometric identities and/or Euler's formula.]
3. Explain why the inner product (on the domain $[-1, 1]$) of an odd function and an even function will be zero.
4. We say that two functions are *orthogonal* if their inner product is zero. Explain why this nomenclature makes sense.

   Describe how the vector space X_D includes an infinite collection of functions which are orthogonal to one another. Compare/contrast to $\mathbb{R}^3$.

Exercise 2.3.3. (Done in class.) In Exercise 2.2.1, we considered the length functional for functions $\mathbf{u}\colon [0,1] \to \mathbb{R}^2$ that represent paths from $\mathbf{a}$ to $\mathbf{b}$.

In Cartesian coordinates (x, y), we write $\mathbf{u} = (u_x, u_y)$ and the functional is

$$L[\mathbf{u}] = L[(u_x, u_y)] = \int_0^1 \sqrt{\left(\frac{du_x}{dt}\right)^2 + \left(\frac{du_x}{dt}\right)^2}\, dt.$$

Let $\mathbf{v} = (v_x, v_y)$ be a test function. Show that

$$\frac{d}{d\varepsilon}\left[L[\mathbf{u} + \varepsilon\mathbf{v}]\right]_{\varepsilon=0} = \int_0^1 \frac{1}{\sqrt{\left(\frac{du_x}{dt}\right)^2 + \left(\frac{du_x}{dt}\right)^2}} \begin{pmatrix} \frac{du_x}{dt} \\ \frac{du_y}{dt} \end{pmatrix} \cdot \begin{pmatrix} \frac{dv_x}{dt} \\ \frac{dv_y}{dt} \end{pmatrix} dt.$$

Use this to deduce the Euler-Lagrange equations for L. Then show that straight paths are critical points of the functional.

Exercise 2.3.4. Repeat the previous exercise in polar coordinates. You'll need to first deduce that for a math given by $(r, \theta) = (u_r, u_\theta)$, the length functional is

$$L[\mathbf{u}] = L[u_r, u_\theta] = \int_0^1 \sqrt{\left(\frac{du_r}{dt}\right)^2 + \frac{1}{u_r^2}\left(\frac{du_\theta}{dt}\right)^2}\, dt.$$

Exercise 2.3.5. In Exercise 2.2.2 you should have found that in spherical coordinates, as given by (2.1), we have

$$dL^2 = dr^2 + r^2 \sin^2 \phi d\theta^2 + r^2 d\phi^2.$$

Thus for a path given by $(r, \theta, \phi) = (1, u_\theta, u_\phi)$ the length functional is

$$L[u_\theta, u_\phi] = \int_0^1 \sqrt{\sin^2(u_\phi)\left(\frac{du_\theta}{dt}\right)^2 + (u_r)^2\left(\frac{du_\phi}{dt}\right)^2}\, dt.$$

Find the corresponding Euler-Lagrange equations. Then show that traversing the equator, and that traversing a meridian, is a solution.

Exercise 2.3.6. In Exercise 2.2.3 you found that the functional for the catenoid problem is

$$A[u] = \int_{x_0}^{x_1} u(x)\sqrt{1 + [u'(x)]^2}\, dx.$$

Find the corresponding Euler-Lagrange equation.

Exercise 2.3.7. In Exercise 2.2.4 you found that the functional for the catenary problem is

$$F[u] = 10\rho \int_{x_0}^{x_1} u(x)\sqrt{1 + [u'(x)]^2}\, dx.$$

Find the corresponding Euler-Lagrange equation.

Exercise 2.3.8. In Exercise 2.2.5 you found that the time functional for the Brachistrochrone problem is

$$F[w] = C\int_{x_0}^{x_1} \frac{\sqrt{1 + [w'(x)]^2}}{w(x)}\, dx$$

1. Find the corresponding Euler-Lagrange equation.
2. Read about the history of the problem:
    - http://www-history.mcs.st-and.ac.uk/HistTopics/Brachistochrone.html

- http://en.wikipedia.org/wiki/Brachistochrone_curve#History

Exercise 2.3.9. In Exercise 2.2.6 you found the following action functionals:

1. Vertical motion under the influence of gravity near the surface of the earth: Here we describe motion by a single function u which represents the height at time t. The action is

$$A[u] = \int_0^T \left\{ \frac{1}{2} \left(\frac{du}{dt} \right)^2 - m\, g\, u \right\} dt$$

2. Frictionless motion under the influence of a spring: Describe the motion by function u, representing the horizontal displacement from equilibrium. The action is

$$A[u] = \int_0^T \left\{ \frac{1}{2} \left(\frac{du}{dt} \right)^2 - \frac{1}{2} k u^2 \right\} dt$$

3. Motion in three dimensions under the influence of gravity of nearby massive object. Here $\mathbf{u}$ represents the position of the particle, in a coordinate system centered at the massive object. The action integral is

$$A[\mathbf{u}] = \int_0^T \left\{ \frac{1}{2} \left\| \frac{d\mathbf{u}}{dt} \right\|^2 - \frac{GM}{\|\mathbf{u}\|} \right\} dt$$

In each of the three cases above, find the Euler-Lagrange equations.

Exercise 2.3.10. In Exercise 2.2.7 you found the free action integral to be

$$A[u] = \int_0^T \frac{1}{2} \left\| \frac{d\mathbf{u}}{dt} \right\|^2 dt.$$

Find the Euler-Lagrange equations.

Exercise 2.3.11. In Exercise 2.2.8 you found the pendulum action to be

$$A[\theta] = \int_0^T \left\{ \frac{1}{2} L^2 \left(\frac{d\theta}{dt} \right)^2 - gL(1 - \cos\theta) \right\} dt,$$

where $\theta = 0$ corresponds to the pendulum being at its lowest point.

FindThoseLaplacians

Exercise 2.3.12. In Exercise 2.2.9 you found the following formulas for the Dirichlet energy functionals:

- In two-dimensional Cartesian coordinates

$$E[u] = \frac{1}{2} \iint \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right\} dx\, dy$$

- In two-dimensional polar coordinates

$$E[u] = \frac{1}{2} \iint \left\{ \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} \right)^2 \right\} r\, dr\, d\theta$$

- In three-dimensional Cartesian coordinates

$$E[u] = \frac{1}{2} \iiint \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right\} dx\, dy\, dz$$

- In three-dimensional spherical coordinates

$$E[u] = \frac{1}{2} \iiint \left\{ \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2 \sin^2\phi} \left(\frac{\partial u}{\partial \theta} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \phi} \right)^2 \right\} r^2 \sin\phi\, dr\, d\theta\, d\phi$$

We now consider functions $u \colon \Omega \to \mathbb{R}$ for various regions Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$, assuming the *Dirichlet boundary condition:* that $u = 0$ along the boundary of Ω.

1. Suppose Ω is the rectangle $[0, L] \times [0, M]$ in $\mathbb{R}^3$.
   (a) What are the limits of integration of the Dirichlet energy functional?

(b) Find the Euler-Lagrange equation.

2. Suppose Ω is the disk of radius R in $\mathbb{R}^2$.
   (a) Explain why the following limits of integration for the Dirichlet energy functional make sense: $0 \leq r \leq R$ and $-\pi \leq \theta \leq \pi$.
   (b) Explain why, for any continuous function u, we have $u(r, -\pi) = u(r, \pi)$ and $\partial_\theta u(r, -\pi) = \partial_\theta u(r, \pi)$. What kind of boundary conditions are these?
   (c) Find the Euler-Lagrange equation.
3. Suppose Ω is the unit cube in the first quadrant of $\mathbb{R}^3$.
   (a) What are the limits of integration of the Dirichlet energy functional?
   (b) Find the Euler-Lagrange equation.
4. Suppose Ω is the solid ball of radius R in $\mathbb{R}^3$.
   (a) What are the limits of integration for the Dirichlet energy functional?
   (b) What type of boundary conditions must hold when when $\theta = \pm\pi$? item Find the Euler-Lagrange equation.

Exercise 2.3.13. Recall the following facts from Calculus III:

- Divergence theorem:

$$\int_\Omega \operatorname{div} \mathbf{f} \, dV = \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} \, dA.$$

- Product rule: If g is a function and $\mathbf{f}$ is a vectorfield, then

$$\operatorname{div}(g\mathbf{f}) = \operatorname{grad} g \cdot \mathbf{f} + g \operatorname{div} \mathbf{f}.$$

1. Explain, briefly, why each of the facts above is true.
2. We now consider the Dirichlet energy functional for functions u, defined on region Ω, which satisfy $u = 0$ along the boundary $\partial\Omega$. Show that

$$\frac{d}{d\varepsilon} \left[E(u + \varepsilon v) \right]_{\varepsilon = 0} = \int_\Omega (\operatorname{grad} u) \cdot (\operatorname{grad} v) \, dV.$$

3. Use the product rule to show that

$$(\operatorname{grad} u) \cdot (\operatorname{grad} v) = \operatorname{div}[v \operatorname{grad} u] - v \operatorname{div}(\operatorname{grad} u).$$

4. Use the fact that v is a test function to conclude that

$$\frac{d}{d\varepsilon} \left[E(u + \varepsilon v)\right]_{\varepsilon=0} = \int_{\Omega} (-\operatorname{div}(\operatorname{grad} u)) v \, dV.$$

   and that the Euler-Lagrange equation for the Dirichlet energy functional is

$$-\operatorname{div}(\operatorname{grad} u) = 0.$$

5. Recall from Calc III that the Laplacian Δ is defined by

$$\Delta u = \operatorname{div}(\operatorname{grad} u).$$

   Compute Δu in Cartesian coordinates.[1]

6. Take a look back at Exercise 2.3.12. Notice that you've just computed the Laplacian in several different coordinate systems!

[1]In physics land, the Laplacian is often denoted $\nabla^2 u$ instead of Δu. In mathematics, the notation $\nabla^2 u$ is usually reserved for the Hessian. *Insert your favorite joke about physicists and the second derivative test here.* Note, however, that mathematicians are divided about whether one should define $\Delta u = \operatorname{div}(\operatorname{grad} u)$, as I have done, or define it to be $-\operatorname{div}(\operatorname{grad} u)$. *Insert awkward joke about mathematicians here.*