2.2 Constructing functionals

PlanarGeodesic

Exercise 2.2.1. (Done in class.) Let $\mathbf{a}$, $\mathbf{b}$ be two points in $\mathbb{R}^2$. Let $\mathbf{X}$ be all (differentiable) functions $\mathbf{u}\colon [0,1] \to \mathbb{R}^2$ such that $\mathbf{u}(0) = \mathbf{a}$ and $\mathbf{u}(1) = \mathbf{b}$.

Write down, in Cartesian coordinates, the expression for the function $L[\mathbf{u}]$ which gives the length of the path described by $\mathbf{u} \in \mathbf{X}$.

Exercise 2.2.2. Repeat Exercise 2.1 with the modification that points $\mathbf{a}$ and $\mathbf{b}$ are on the unit sphere S^2 inside $\mathbb{R}^3$, and the path given by $\mathbf{u}$ must travel along the surface of the sphere.

Please use the following spherical coordinate convention:

$$\begin{aligned} x &= \cos\theta \, \sin\phi \\ y &= \sin\theta \, \sin\phi \\ z &= \cos\phi \end{aligned} \tag{2.1}$$

SphericalConvention

[**Warning:** Conventions vary about which angle is θ and which is ϕ, as well as where certain angles are zero. Be careful and pay attention!]

Exercise 2.2.3. Suppose $u\colon [x_0, x_1] \to \mathbb{R}$ is a positive function whose graph passes through the points (x_0, y_0) and (x_1, y_1). Find the functional which represents the area of the surface obtained by rotating the graph of u about the x-axis.

The minimizing surface joining the two circles is called a *catenoid*.

Exercise 2.2.4. (Done in class) Suppose a chain is hanging between two pullies. [Insert picture here.] Assuming a constant linear mass density, find the function which measures the potential energy of the chain's configuration.

The functional should be equivalent to that for the catenoid; the optimizing curve is called a *catenary*.

Exercise 2.2.5. Suppose $u\colon [x_0, x_1] \to \mathbb{R}$ is a function whose graph passes through the points (x_0, y_0) and (x_1, y_1). The goal is to find u such that a bead sliding under the force of gravity down a wire whose shape is given by u does so in the least time possible. Let s represent distance along the curve and let $v(s)$ be the velocity of the bead at distance s from the point (x_0, y_0). Argue that the total time is

$$T = \int_0^L \frac{1}{v(s)}\, ds, \tag{2.2}$$

where L is the (unknown) length of the path.

Conservation of energy for the system requires that

$$\frac{1}{2}mv^2 + mgu \tag{2.3}$$

be constant. Here m is the mass of the bead and g is gravitational force on the surface of the earth. Show that we may write

$$v(x) = \sqrt{\frac{2c}{m} - 2gu(x)} \tag{2.4}$$

for some constant c, which implies

$$T[u] = \int_{x_0}^{x_1} \sqrt{\frac{1 + \left(\frac{du}{dx}\right)^2}{\frac{2c}{m} - 2gu(x)}}\, dx. \tag{2.5}$$

Change variables to $w(x) = \frac{1}{2g}\left(\frac{2c}{m} - 2gu(x)\right)$ to obtain

$$T[w] = (\text{constant}) \int_{x_0}^{x_1} \sqrt{\frac{1 + \left(\frac{dw}{dx}\right)^2}{w(x)}}\, dx. \tag{2.6}$$

The optimizing curve is called a *Brachistochrone.*

Exercise 2.2.6. *Action.* (Done in class.) If a particle of unit mass is moving along trajectory $\mathbf{u}(t)$ for $0 \le t \le T$, then the action integral is given by

$$A[\mathbf{u}] = \int_0^T (K - U)\; dt,$$

where $K = \frac{1}{2}\|\frac{d\mathbf{u}}{dt}\|^2$ is the kinetic energy and $U = U(\mathbf{u})$ is the potential energy.

Construct the action integrals for:

1. vertical motion under the influence of gravity near the surface of the earth.
2. frictionless horizontal motion under influence of a spring.
3. motion in three dimensions under influence of gravity of nearby massive object.

Exercise 2.2.7. *Free motion.* Suppose a unit mass moving along trajectory $\mathbf{u}(t)$ is not subject to any potential energy. What is the action integral?

Exercise 2.2.8. *Pendulum* Consider a pendulum consisting of a rigid rod (which we assume to be of insignificant mass) of length L, with a unit mass at one end. Write down an action integral for the mass, with potential energy coming from gravity at the surface of the earth.

Exercise 2.2.9. *Dirichlet energy* The Dirichlet energy of a function $u : \mathbb{R}^n \to \mathbb{R}$ is given by

$$E[u] = \frac{1}{2}\int_{\mathbb{R}^n} \|\nabla u\|^2 \, dV, \tag{2.7}$$

where dV is the area element in $\mathbb{R}^2$, the volume element in $\mathbb{R}^3$, etc.

1. Write down the Dirichlet energy for a function $u : \mathbb{R}^2 \to \mathbb{R}$ in both Cartesian and polar coordinates.
2. Write down the Dirichlet energy for a function $u : \mathbb{R}^3 \to \mathbb{R}$ in both Cartesian and spherical coordinates. Use the spherical coordinate convention (2.1).