Chapter 1

Introduction

1.1 What is the course about?

- The main theme of this course is the *study of functions*.
- *Calculus* provides a great collection of tools for studying functions. We can find the various rates of change of a function, compute the cumulative effect of a function, approximate a function by a polynomial, etc. We can do this in both one and higher dimensions.
- Most of what we have done in the past is study one function at a time. In this class we study the structure of *spaces of functions*. Part of this involves constructing certain collections of functions called *special functions*.
- In order to apply tools of calculus to a function, it needs to be 'sufficiently nice.' Thus we spend some time describing various collections (called *spaces* or *classes*) of functions.
- Not surprisingly, functions which are considered "special" in the calculus sense are those which satisfy certain differential equations. Thus we will end up using certain differential equations as a tool for studying functions.

- It also turns out to be very helpful to draw an analogy between vectors in $\mathbb{R}^3$ and certain classes of functions – thus we study *vector spaces* of functions.

- A great deal of the course turns out to be related to the idea of *ways to represent functions*.

  What might this mean? At its heart, a function is a systematic and consistent way for assigning an output in a codomain to each element in a domain. (If you have taken our Discrete course, you have learned to express this as a certain type of relation on the product of the domain and codomain.) In some circumstances, one has a (more or less) explicit formula which can be used to make this assignment, but in most circumstances this is not the case. (We may only know from some abstract result, such as the fundamental theorem of ODE, that the function exists.) By 'representation of a function' we mean a description of the rule of assignment given in terms of other objects which are 'well-known.'

  For example: One might deduce somehow that there is a function $x \mapsto \ln x$, but there is no 'easy' way to write down a formula for this function. On the domain $(-1, 1)$ we can, however, represent the function using the power series representation

  $$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, x^k \tag{1.1}$$

  NaturalLogPowerSeries

- Finally, it is helpful to have some motivations about why certain functions are useful, or where such functions arise. As the previous discussion indicates, one place where functions naturally arise is in studying ODEs. There is a well-trodden path: physical system $\rightsquigarrow$ differential equation $\rightsquigarrow$ function (which solves equation).

1.2 The exponential function

- Old friend e^x
- One way to view: There is a number e and we are raising it to power of x... whatever that means
- Another perspective: start with basic growth model

$$\frac{dP}{dt} = rP \tag{1.2}$$

SimpleBasicGrowthMode

Here r is some constant

- We use the following:

ACPowerSeries

Principle (Absolutely convergent power series)**.** *Suppose the power series* $\sum_k a_k (x - x_*)^k$ *converges absolutely for certain values of* x*.*

1. *The power series defines a function on a domain consisting of all* x *for which the series converges.*
2. *The function defined by the power series is the zero function if and only if each coefficient* a_k *vanishes.*
3. *We are allowed to differentiate/integrate the function term-by-term.*
4. *We are allowed to do other algebraic operations term-by-term.*

The proof of this principle is postponed to the real analysis course.

- Look for solution of form $P = \sum_{k=0}^{\infty} a_k x^k$. (That is, take $x_* = 0$.)
- We find that there is no restriction on the first coefficient a_0
- The other coefficients satisfy a recursion relation

$$a_k = rka_{k-1} \tag{1.3}$$

ExponentialRecursionR

- Thus we see that in order to get a solution we need $a_k = a_0 \frac{r^k}{k!}$
- Therefore we want to construct a function by

$$P = a_0 \sum_{k=0}^{\infty} \frac{1}{k!}(rx)^k$$

- We now need to check that the coefficients actually give us a function. In particular, we need to check that the power series converge. Here we can use the ratio test:

Principle (Ratio test)**.** *The sum $\sum_k a_k x^k$ converges absolutely if*

$$\lim_{k\to\infty} \left| \frac{a_{k+1}x^{k+1}}{a_k x^k} \right| < 1.$$

- This tells us that we get a function for all values of x
- Once we have absolute convergence we can verify that the function satisfies the differential equation.

  We then define the exponential function as the solution with $r = 1$ and $P(0) = 1$
- Of course, we also want all the properties of the exponential function:

$$e^a e^b = e^{a+b} \tag{1.4}$$

$$(e^a)^b = e^{ab} \tag{1.5}$$

  We outsource this to Real Analysis class.

Exercise 1.2.1.

1. Show that

$$(1 - x)(1 + x + x^2 + \cdots + x^n) = 1 - x^{n+1}.$$

Use the result to find a nice formula for

$$1 + x + x^2 + \cdots + x^n$$

2. Use the formula you found above to find a power series for the function

$$\frac{1}{1-x}.$$

For which values of x does the power series converge?

3. Integrate to obtain (1.1).

4. Find a power series for the function

$$\frac{1}{1+9x^2}.$$

For which values of x does it converge?

Exercise 1.2.2. Here's another way to think about the natural logarithm function – as the inverse to the exponential function.

1. Suppose that $l(x)$ is the function such that $\exp[l(x)] = x$, whenever $l(x)$ is defined. Show that the chain rule implies that $l(x)$ satisfies the differential equation

$$\begin{aligned} l'(x) &= \frac{1}{\exp[l(x)]} = \frac{1}{x} \\ l(1) &= 0. \end{aligned}$$

2. We now suppose that $l(x)$ has a power series expansion centered at $x_* = 1$, meaning that

$$l(x) = \sum_{k=0}^{\infty} a_k (x-1)^k.$$

Explain why $a_0 = 0$ and that we may assume the sum starts with $k = 1$.

3. Write $\frac{1}{x} = \frac{1}{1-[-(x-1)]}$; then use the geometric series to construct a series

expansion for $\frac{1}{x}$ centered at $x_* = 1$. For which values of x does the series converge?

4. Using the power series expansions for $l(x)$ and for $\frac{1}{x}$, together with the differential equation for $l(x)$, determine the coefficients a_k.

5. Conclude by forming a power series for $l(x)$ and comparing to (1.1).

1.3 The simple harmonic oscillator

We now introduce the most important example of the course: SHO.

- Math 235 derivation: Hooke & Newton leads to

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{1.6}$$

SHO-Omega

  where $\omega = \sqrt{\frac{k}{m}}$.

- Write as a first order system

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\omega^2 x \end{aligned} \tag{1.7}$$

SHO-Omega-FirstOrderS

- Recall that this is a special case of Hamiltonian systems:

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -U'(x) \end{aligned} \tag{1.8}$$

  where $U(x)$ is the potential function and

$$\begin{aligned} H &= \frac{1}{2}v^2 + U(x) \\ &= \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + U(x) \end{aligned} \tag{1.9}$$

  is a conserved quantity.

- Questions:

  1. Where does SHO come from? Meaning, is there a way to get it from "first principles"?
  2. How to build solutions "from scratch"?

  Main idea: *Find an approach that generalizes to other situations!*

Exercise 1.3.1.

1. Write (1.7) using vector-matrix notation.
2. Use the "eigenstuff" method from Math 235 to find the general solution.
3. Finally, write down the general solution to the first-order equation (1.6).

Exercise 1.3.2. Recall the method of understanding Hamiltonian systems by using "energy diagrams." Draw the energy diagrams for the following potentials and describe the corresponding behavior of solutions:

1. $U(x) = x^3 - x^2$
2. $U(x) = \cos x$

Exercise 1.3.3. Suppose we didn't know that the cosine and sine functions existed. Here we show how to discover them using (1.6).

1. We suppose that (1.6) has a solution with a power series, namely that

$$x(t) = \sum_{k=0}^{\infty} a_k t^k.$$

   Plug this in to (1.6) and determine a recursion relation amongst the a_k.

2. "Notice" that the even and odd coefficients are independent, in the sense that specifying a_0 tells you a_2, which tells you a_4, etc. Similarly, specifying a_1 tells you a_3, etc. Express this by constructing recursion relations of the form

$$a_{2l} = \boxed{\text{ stuff involving } l,\ \omega, \text{ and } a_{2(l-1)}. }$$

$$a_{2l+1} = \boxed{\text{ stuff involving } l,\ \omega, \text{ and } a_{2(l-1)+1}. }$$