3.3 Wave equation on the disk, part 1

Exercise 3.3.1. In class we derived a program for finding solutions to the wave equation on the unit disk. The program involved finding functions $\Theta(\theta)$, $R(r)$, and $A(t)$.

1. Explain the logic and process of this procedure for finding solutions to the wave equation. Your explanation should include the equations that each of these functions need to satisfy, as well as the logical dependencies between the various equations. (That is, which things depend on what.)
2. We found a long list of suitable functions $\Theta(\theta)$. What are they?
3. In the case that $\lambda = 0$, the eigenvalue problem for the Laplacian reduces to *Laplace's equation*:

   $$\Delta\psi = 0.$$

   We found a long list of functions $\psi(r, \theta)$ satisfying Laplace's equation. What are they?
4. What needs to happen next if we are going to continue this program for finding solutions to the wave equation?

3.4 Laplace equation on the disk

Exercise 3.4.1. In class, we found a large number of solutions to Laplace's equation in polar coordinates.

1. Which of those solutions are defined everywhere on the disk?
2. Restricting attention to those solutions, what is the general solution to Laplace's equation on the disk?
3. Let $u(r, \theta)$ be the general solution. Explain why we can prescribe the value of u along the boundary (at $r = 1$), and obtain a solution to Laplace's equation.

4. Fix some function f, defined on the boundary of the unit disk, and suppose we have two solutions to the following BVP:

$$\begin{aligned} \Delta u &= 0 \qquad \text{on unit disk} \\ u &= f \qquad \text{on boundary.} \end{aligned}$$

Show that the function w satisfies the BVP

$$\begin{aligned} \Delta w &= 0 \qquad \text{on unit disk} \\ w &= 0 \qquad \text{on boundary.} \end{aligned}$$

Then show that

$$0 = \int_\Omega w \Delta w \, dA = -\int_\Omega \|\nabla w\|^2 \, dA.$$

Conclude that $w = 0$ and hence that $u_1 = u_2$. You've just shown uniqueness of solutions!

5. Solve the following BVP:

$$\begin{aligned} \Delta u &= 0 \qquad \text{on unit disk} \\ u &= f \qquad \text{on boundary.} \end{aligned}$$

where the function f is given by

$$f(\theta) = \begin{cases} 1 & \text{if } 0 < \theta \leq \pi \\ -1 \text{ if } -\pi < \theta \leq 0 \end{cases}$$

Use Mathematica to obtain an (approximate) plot of the solution.