TOPIC 5

First steps in summation

Main ideas.

- Summation notation
- Geometric sums
- Telescoping sums

Exercises.

Exercise 5.1. Write the following sums in summation notation:

(1) $1 + 4 + 9 + 16 + \cdots + 121$
(2) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{15}{16}$
(3) $1 + 3 + 9 + 27 + \cdots + 243$
(4) $3 + 6 + 12 + 24 + \cdots + 96$
(5) $1 - 2 + 3 - 4 + \cdots - 18$
(6) $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{27}$
(7) $\frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \frac{3}{625}$
(8) $\frac{6}{5} + \frac{12}{25} + \frac{24}{125} + \frac{96}{625}$

Solution:

(1) $\displaystyle\sum_{k=1}^{11} k^2$
(2) $\displaystyle\sum_{k=1}^{15} \frac{k}{k+1}$
(3) $\displaystyle\sum_{k=0}^{5} 3^k$
(4) $\displaystyle\sum_{k=0}^{5} 3 \cdot 2^k$

Exercise 5.2. Find formulas / expressions for the following sums:

(1) $\displaystyle\sum_{k=1}^{17} k^2$
(2) $\displaystyle\sum_{k=1}^{N} k^2$
(3) $\displaystyle\sum_{k=3}^{12} k^3$
(4) $\displaystyle\sum_{k=3}^{N} k^3$
(5) $\displaystyle\sum_{n=1}^{25} \frac{1}{n(n+1)}$
(6) $\displaystyle\sum_{n=1}^{N} \frac{1}{n(n+2)}$
(7) $\displaystyle\sum_{n=2}^{42} \frac{1}{n^2-1}$
(8) $\displaystyle\sum_{n=2}^{N} \frac{1}{n^2-1}$
(9) $\displaystyle\sum_{k=0}^{7} \frac{1}{3^k}$
(10) $\displaystyle\sum_{k=0}^{N} \frac{2^k}{3^k}$
(11) $\displaystyle\sum_{k=0}^{12} (-1)^k \frac{2^k}{3^k}$
(12) $\displaystyle\sum_{k=0}^{n} \frac{5^k}{3^k}$

Solution:

(1) $\displaystyle\sum_{k=1}^{17} k^2 = \frac{(17)(17+1)(2\cdot 17+1)}{6} = 1785$

(2) $\displaystyle\sum_{k=1}^{N} k^2 = \frac{(N)(N+1)(2\cdot N+1)}{6}$

(5) $\displaystyle\sum_{n=1}^{25} \frac{1}{n(n+1)} = \sum_{n=1}^{25} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{1}{1} - \frac{1}{26} = \frac{25}{26}$

(7) $\displaystyle\sum_{n=2}^{42} \frac{1}{n^2-1} = \sum_{n=2}^{42} \frac{1}{2}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] = \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} - \frac{1}{42} - \frac{1}{43}\right]$

(12) $\displaystyle\sum_{k=0}^{n} \frac{5^k}{3^k} = \sum_{k=0}^{n} \left(\frac{5}{3}\right)^k = \frac{1-\left(\frac{5}{3}\right)^{n+1}}{1-\left(\frac{5}{3}\right)} = -\frac{3}{2}\left[1 - \left(\frac{5}{3}\right)^{n+1}\right] = \frac{3}{2}\left(\frac{5}{3}\right)^{n+1} - \frac{3}{2}$