TOPIC 4

Sequences

Main ideas.

- Functions & sequences
- Features of functions and sequences
- Limits & convergence for sequences

Exercises.

Exercise 4.1. The point of this problem is for you to become familiar with some vocabulary surrounding calculus.

(1) Describe the technical meaning of the words *function*, *domain*, *graph* as they appear in the context of this course.
(2) What is the difference between the concepts *function* and *equation*? Give an illustrative example.
(3) What is a *sequence*? How does this concept relate to that of a function?
(4) Compare and contrast the concepts *discrete* and *continuous*.
(5) Give examples of both continuous functions and (discrete) sequences which are...

    ...increasing.
    ...decreasing.
    ...bounded from above.
    ...bounded from below.
    ...unbounded.

(6) Give both a "careful" definition of the limit of a sequence, and also an "intuitive" description of the limit. Illustrate with an example.
(7) Give three examples of convergent sequences and three examples of divergent sequences.

Solution:

(1) A *function* provides a correspondence between a collection of inputs, called the *domain* and outputs (which are in the codomain or range space). The *graph* of a function is a plot of pairs of points, the first of which is in the

domain of a function and the second of which is the corresponding output. Typically we put the domain on the horizontal axis.

(2) A function is a multi-parted object, consisting of a domain, a range, and a rule that gives one element of the range for each element in the domain. An equation is obtained by requiring two mathematical expressions to be equal to one another.

An example of a function is the squaring function. We take the domain to be all real numbers, the range to be real numbers, and for each input x, the output of the function is x^2. Sometimes we write this function as $f(x) = x^2$.

An example of an equation is $x^2 - 3x + 2 = 0$.

(3) A sequence is an ordered list of numbers, which we may view as a function with domain $\{0, 1, 2, 3, \dots\}$ or $\{1, 2, 3, \dots\}$.

(4) Continuous-type functions have intervals as part of their domain, thus these functions are defined as the input varies continuously from one to another. In contrast, discrete-type functions have "isolated" points in their domain; a typical domain is $\{1, 2, 3, \dots\}$.

(5) Examples may vary...

(6)

Careful definition: We say that $a_k \to L$ if for every $\epsilon > 0$ we can find N so that

$$L - \epsilon < a_k < L + \epsilon$$

when $k \geq N$.

Intuitive definition: We way that $a_k \to L$ if for every small interval surrounding L we can find a point along the sequence so that all following numbers are in the small interval.

Examples may vary...

(7) Examples will vary...

Exercise 4.2. For each sequence below, do the following:

- Write out the first few terms of the sequence.
- Plot the first few terms of the sequence.
- Decide if the sequences is increasing/decreasing, bounded/unbounded, etc. Give rationale when possible.
- Determine if the sequence is convergent or divergent. Provide the best rationale you can.

(1) $a_n = \frac{(-1)^n}{n}$

(2) $a_n = \cos(n\pi)$

(3) $a_n = \left(1 + \frac{1}{n}\right)^n$

(4) $a_n = \ln(2n + 1) - \ln(n)$

(5) $a_0 := 1, \quad a_{n+1} := \frac{1}{2}\left(a_n + \frac{4}{a_n}\right)$

(6) $a_0 := 1, \quad a_{n+1} := (n + 1) \cdot a_n$

(7) $a_0 := 0, \quad a_{n+1} := a_n + \frac{1}{2^n}$

(8) $a_n = \frac{1}{n!}$

(9) $a_n = \frac{e^n}{n!}$

(10) $a_0 := 0, \quad a_1 := 1, \quad a_{n+1} := 5a_n - 6a_{n-1}$

(11) $a_n := \int_{-\pi}^{\pi} x\sin(nx)\,dx$

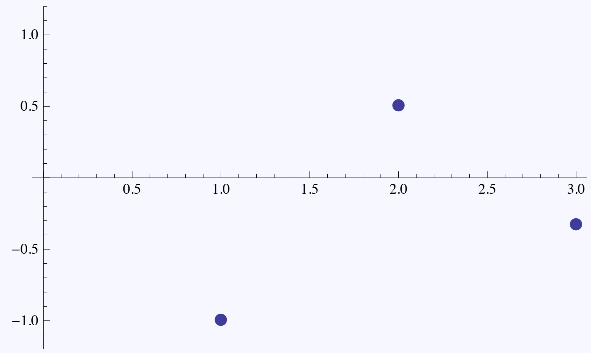
(12) a_n is the value of the Riemann sum for the function $f(x) = x^2$ over the interval $[0, 1]$ obtained by using n subdivisions and right endpoints as sample points.

Solution:

(1) We compute

$$a_1 = -1$$
$$a_2 = \frac{1}{2}$$
$$a_3 = -\frac{1}{3}$$

The plot of these is

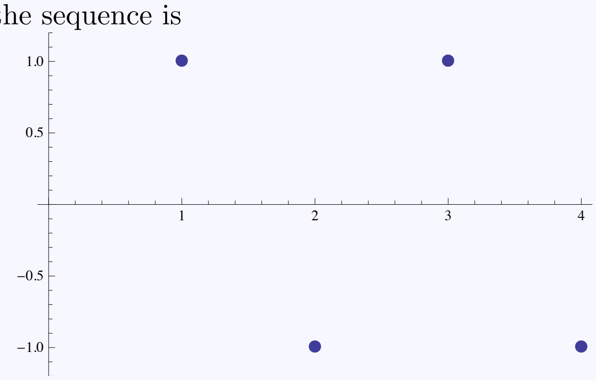


The sequence is neither increasing, nor decreasing. It is, however, bounded and converges to zero because the numbers are clearly getting smaller in magnitude.

(2) We compute

$$a_0 = 1$$
$$a_1 = -1$$
$$a_2 = 1$$
$$a_3 = -1$$

The plot of the sequence is

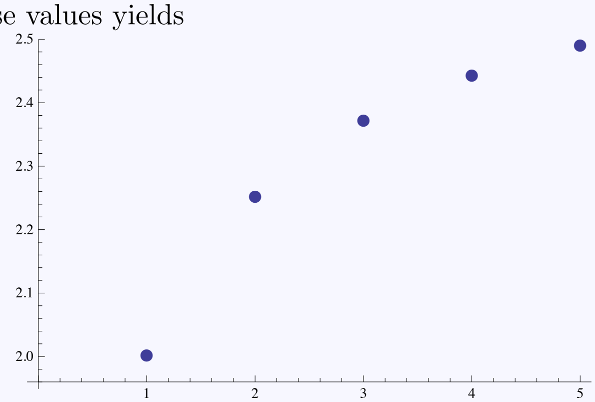


The sequence is neither increasing nor decreasing. It is bounded both above and below, but does not converge. This is because no matter how far out the list we go, we still oscillate between 1 and -1.

(3) We compute

$$a_1 = 2$$
$$a_2 = \frac{9}{4}$$
$$a_3 = \frac{64}{27} \approx 2.37037$$
$$a_4 = \frac{625}{256} \approx 2.44141$$
$$a_4 = \frac{7776}{3125} \approx 2.48832$$

Plotting these values yields

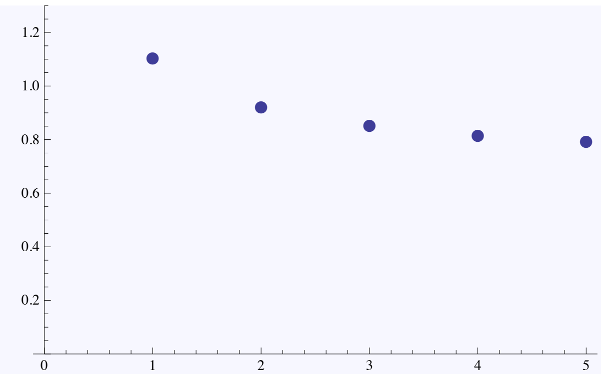


Based on the plot, it seems like the sequence is increasing and is thus bounded from below. The sequence is also bounded above. In fact, the sequence converges to e, which you might remember from an earlier course...

(4) We compute

$$a_1 = 1.09861,$$
$$a_2 = 0.916291,$$
$$a_3 = 0.847298,$$
$$a_4 = 0.81093,$$
$$a_5 = 0.788457$$

which gives us the plot



The sequences is decreasing and bounded both above and below. We can see that the sequence converges to $\ln 2$, which can be seen by observing that

$$a_n = \ln\left(\frac{2n+1}{n}\right)$$

and that the argument of the log function $\frac{2n+1}{n} = 2 + \frac{1}{n} \to 2$.

Exercise 4.3.

(1) Suppose we define the sequence S_n by $S_n = 1 + 2 + 3 + \cdots + n$. Do you think S_n converges as $n \to \infty$? Explain.

(2) Suppose now that we define a new sequence $T_n = \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}$. Do you think T_n converges as $n \to \infty$? Explain.

(3) Next, define R_n by $R_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$. Do you think R_n converges as $n \to \infty$? Explain.

(4) Show (using Gauß' trick) that $S_n = \frac{n(n+1)}{2}$. Use this to "check" your previous responses.

Exercise 4.4. Define the sequence S_n by $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$.

(1) Find a formula for S_n. [Hint, it is always the case that $S_n < 1$.]
(2) Use your formula to conclude that S_n converges. What does it converge to?

Solution:

We compute

$$
\begin{aligned}
S_1 &= \frac{1}{2} = \frac{2^1 - 1}{2^1} \\
S_2 &= \frac{3}{4} = \frac{2^2 - 1}{2^2} \\
S_3 &= \frac{7}{8} = \frac{2^3 - 1}{2^3} \\
S_4 &= \frac{15}{16} = \frac{2^4 - 1}{2^4} \\
&\vdots \\
S_n &= \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.
\end{aligned}
$$

Since 2^n gets large as n gets large, we see that $S_n \to 1$ as $n \to \infty$.

Exercise 4.5. Suppose we have a sequence a_n such that $a_n \to 0$ as $n \to \infty$.

(1) Describe the behavior of the sequence $b_n = \dfrac{1}{a_n}$ as $n \to \infty$? Explain your reasoning.
(2) Describe the behavior of the sequence $b_n = \sin(a_n)$ as $n \to \infty$? Explain your reasoning.
(3) Describe the behavior of the sequence $b_n = e^{a_n}$ as $n \to \infty$? Explain your reasoning.
(4) Describe the behavior of the sequence $b_n = \dfrac{a_n}{3 + a_n}$ as $n \to \infty$? Explain your reasoning.

Solution:

(1) Since a_n is getting close to zero, $\frac{1}{a_n}$ is getting very large, though it could be either large positive or large negative. In either case, we have b_n diverging.
(2) Since a_n is getting close to zero, we have $\sin a_n$ is getting close to zero as well. Thus we say that b_n is converging towards zero.
(3) Since a_n is getting close to zero, we have e^{a_n} is getting close to 1. Thus we say that b_n is converging towards 1.
(4) When a_n is close to zero, the numerator is close to zero while the denominator is close to 3. Thus $b_n \to 0$ as n gets large.

Exercise 4.6. Suppose we have a sequence a_n such that $a_n \to +\infty$ as $n \to \infty$.

(1) Describe the behavior of the sequence $b_n = \dfrac{1}{a_n}$ as $n \to \infty$? Explain your reasoning.

(2) Describe the behavior of the sequence $b_n = \dfrac{2a_n + 3}{a_n}$ as $n \to \infty$? Explain your reasoning.

(3) Describe the behavior of the sequence $b_n = \dfrac{2a_n}{3 + a_n}$ as $n \to \infty$? Explain your reasoning.

Solution:

(1) Since a_n is getting large, $\frac{1}{a_n}$ is getting small. Thus $b_n \to 0$ as $n \to \infty$.

(2) We re-write the formula for b_n as

$$b_n = 2 + \frac{3}{a_n}.$$

Since a_n gets large, the second term becomes small and we have $b_n \to 2$ as $n \to \infty$.

(3) We re-write the formula for b_n as follows:

$$b_n = \frac{6 + 2a_n}{3 + a_n} - \frac{6}{3 + a_n} = 2 - \frac{3}{3 + 2a_n}.$$

Since a_n is getting large, we have $3 + 2a_n$ is getting large. Thus $\frac{6}{3+a_n}$ is getting small and we have $b_n \to 2$ as $n \to \infty$.