TOPIC 3

Taylor polynomials

Main ideas.

- Linear approximating functions: Review
- Approximating polynomials
- Key formulas:

$$P_n(x) = a_0 + a_1(x - x_*) + \cdots + a_n(x - x_*)^n \qquad \text{where} \quad a_k = \frac{f^{(k)}(x_*)}{k!}$$
$$P_n(x_* + \Delta x) = a_0 + a_1(\Delta x) + \cdots + a_n(\Delta x)^n$$

Mathematica code. Here is some basic mathematica code for plotting functions.

(1) Plot the function $f(x) = \sin(3x)$.

```
Plot[Sin[3 x], {x, -Pi, Pi}]
```

(2) Plot the function $f(x) = \sin(3x)$ in red, and the line $y = 3x$ in blue.

```
Plot[{Sin[3 x], 3 x}, {x, -Pi, Pi}, PlotRange -> {-3, 3},
PlotStyle -> {Red, Blue}]
```

Here is some more sophisticaled Mathematica code for exploring Taylor series.

(1) The exponential function $f(x) = e^{2x}$

```
Manipulate[
 Plot[{Exp[2 x], Sum[((2 x)^k)/(k!), {k, 0, n}]},
 {x, -1, 1}, PlotRange -> {0, 4},
 PlotStyle -> {{Thick, Red}, {Thick, Blue, Dashed}}],
 {n, 1, 10, 1}]
```

(2) The function $f(x) = \cos x$. (Note that in this case `n` does not correspond to the degree of the polynomial!

```
Manipulate[
 Plot[{Cos[x], Sum[((-1)^k (x)^(2 k))/((2k)!), {k, 0,n}]},
 {x, -3 Pi, 3 Pi},  PlotRange -> {-2, 2},
 PlotStyle -> {{Thick, Red}, {Thick, Blue, Dashed}}],
 {n, 1, 10, 1}]
```

Exercises.

Exercise 3.1. Write down the fourth-order Taylor polynomials, centered at $x_* = 0$, for the following functions:

(1) e^x
(2) $\ln(1+x)$
(3) $\sqrt{1+x}$
(4) $\dfrac{1}{1+x}$

Can you find expressions for the "general" polynomials $P_n(x)$?

Solution:

(1) $e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$
(2) $\ln(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$
(3) $\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{3}{2^3 3!}x^3 - \frac{3 \cdot 5}{2^4 4!}x^4$
(4) $\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + x^4$

Optional: The general formulae for the n^{th} order approximations are

(1)
$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n$$

(2)
$$\ln(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^n \frac{1}{n}x^n$$

(3)
$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{3}{2^3 3!}x^3 - \cdots + (-1)^{n+1}\frac{(1)(3)\dots(2n-3)}{2^n\, n!}x^n$$

(4)
$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + x^4 + \cdots + (-1)^n \frac{1}{n}x^n$$

Exercise 3.2.

(1) Consider $f(x) = \sin x$. Show that $f^{(k)}(0) = 0$ when k is even, and that when $k = 2l + 1$ is odd we have $f^{(2l+1)}(0) = (-1)^l$. Conclude that the Taylor polynomial, centered at $x_* = 0$, for $\sin x$
$$x + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^l}{(2l+1)!}x^{2l+1} + \dots$$
(2) Show that the Taylor polynomial for the cosine function, centered at $x_* = 0$, is given by
$$1 + \frac{-1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{(-1)^l}{(2l)!}x^{2l} + \dots$$
(3) Suppose f is an *even function*, meaning that $f(x) = f(-x)$ for all x. Show that this implies that $f^{(k)}(0) = 0$ whenever k is odd. [Hint: Take k derivatives

of the identity defining even-ness.] What can you conclude about the Taylor polynomials centered at $x_* = 0$?

Exercise 3.3. Using the polynomials you found in Exercises 3.1 and 3.2, find the following Taylor polynomials. [Do not do any extra work!]

(1) Find the fourth-order Taylor polynomial for e^{3x}, centered at $x_* = 0$.
(2) Find the fourth-order Taylor polynomial for $\ln(x)$, centered at $x_* = 1$.
(3) Find the fourth-order Taylor polynomial for $\cos(5x)$, centered at $x_* = 0$.
(4) Find the eighth-order Taylor polynomial for $\sqrt{1+4x^2}$, centered at $x_* = 0$.

Solution:

(1)

$$\begin{aligned} e^{3x} &\approx 1 + 3x + \frac{1}{2}(3x)^2 + \frac{1}{6}(3x)^3 + \frac{1}{24}(3x)^4 \\ &= 1 + 3x + \frac{9}{2}x^2 + \frac{27}{6}x^3 + \frac{81}{24}x^4 \end{aligned}$$

(2) We rewrite $\ln(x) = \ln(1 + [x-1])$. Thus

$$\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

(3)

$$\begin{aligned} \cos(5x) &\approx 1 + \frac{-1}{2!}(5x)^2 + \frac{1}{4!}(5x)^4 \\ &= 1 - \frac{25}{2!}x^2 + \frac{625}{4!}x^4 \end{aligned}$$

(4)

$$\begin{aligned} \sqrt{1+4x^2} &\approx 1 + \frac{1}{2}(4x^2) - \frac{1}{4}(4x^2)^2 + \frac{3}{2^3 3!}(4x^2)^3 - \frac{3\cdot 5}{2^4 4!}(4x^2)^4 \\ &= 1 + \frac{4}{2}x^2 - \frac{16}{4}x^4 + \frac{3\cdot 4^3}{2^3 3!}x^6 - \frac{3\cdot 5\cdot 4^4}{2^4 4!}x^8 \end{aligned}$$

Exercise 3.4. In this exercise, we explore the Taylor polynomial approximation for $f(x) = \frac{1}{1-x}$.

(1) Compute several derivatives of f and conclude that $f^{(k)}(0) = k!$.
(2) Write down the formula for the Taylor polynomial $P_n(x)$ for f, centered at $x_* = 0$.
(3) What is $P_n(1)$? (This will depend on n, of course.) What happens to $P_n(1)$ as n gets large?
(4) What is $f(1)$? In what sense does this "agree" with your response to the previous question?
(5) Find a formula for $P_n(-1)$. (Again, your formula will depend on n.) What happens as n gets large?
(6) What is $f(-1)$? In what sense does this "agree" with your response to the previous question?

(7) Compare $P_n(2)$ with $f(2)$ as n gets large. Are the values close to one another? Explain.

(8) Use the following code to compare the Taylor polynomials to f.

```
Manipulate[
 Plot[{1/(1 - x), Sum[x^k, {k, 0, n}]},
{x, -2, 3}, Exclusions -> {1},PlotRange -> {-5, 5},
PlotStyle -> {{Thick,Red}, {Thick,Blue,Dashed}}],
 {n, 1, 10, 1}]
```

To what extent do the polynomials do a good job approximating the function; to what extent do the polynomials do a poor job?

Solution:

(2) $P_n(x) = 1 + x + x^2 + \cdots + x^n$

(3) $P_n(1) = 1 + 1 + 1 + \cdots + 1 = n$. As n gets large, so does $P_n(1)$.

(4) $f(1)$ is not defined; the function has an asymptote there. This agrees with the result from part (3) in the sense that there is no number $P_n(1)$ approaches as n gets large.

(5) We have

$$P_n(-1) = 1 - 1 + 1 - 1 + 1 - \cdots + (-1)^n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

As n gets large, $P_n(-1)$ oscillates between zero and one.

(6) $f(-1) = \frac{1}{2}$. In some sense this "agrees" with the polynomial approximation, in that $\frac{1}{2}$ is the average of 0 and 1; however, it does not agree in the sense that the approximation never gives values close to the value of the function.

(7) $P_n(2) = 1 + 2 + 4 + \cdots + 2^n$ gets very large as n gets large. On the other hand, $f(2) = -1$. These two are very different.

(8) The polynomials seem to do a good job approximating f in the interval $(-1, 1)$ and otherwise do not do a good job approximation f.

Exercise 3.5. In the previous problem, we saw that the Taylor polynomials $P_n(x)$ approximated the function $f(x)$ very well for a certain range of x, but did not do a good job approximating the function for other x. One might conjecture that this is because the function f considered has an asymptote. In this exercise we see that the Taylor polynomials might not do a good job approximating a function, *even if that function has no vertical asymptotes!*

(1) Consider the function $g(x) = \dfrac{1}{1+x^2}$. Use Exercise 3.1 to show that the Taylor polynomial approximating g near $x_* = 0$ are given by

$$P_n(x) = 1 - x^2 + x^4 - x^6 + \ldots$$

(2) Use the following code to explore how well the Taylor polynomials approximate g. Write a few sentences explaining your findings.

```
Manipulate[
 Plot[{1/(1 + x^2), Sum[(-x^2)^k, {k, 0, n}]},
 {x,-2,3}, Exclusions->{1}, PlotRange->{-1, 2},
 PlotStyle->{{Thick,Red}, {Thick,Blue,Dashed}}],
  {n, 1, 10, 1}]
```

Solution:

(1) Using Exercise 3.1 we have

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} \\ &\approx 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots + (-x^2)^n \\ &= 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n}.\end{aligned}$$

(2) Using the Mathematica code we see that on the interval $(-1, 1)$ the Taylor polynomials approximate the function very well. However, for $x > 1$ and $x < -1$, the Taylor polynomials do not seem to do a good job of approximating the function.