TOPIC 1

Calculus – a perspective

Main ideas.

- Introduction: Calculus as the study of functions
- Approximation: The main idea of calculus
- Simple examples: Piecewise constant and piecewise linear functions

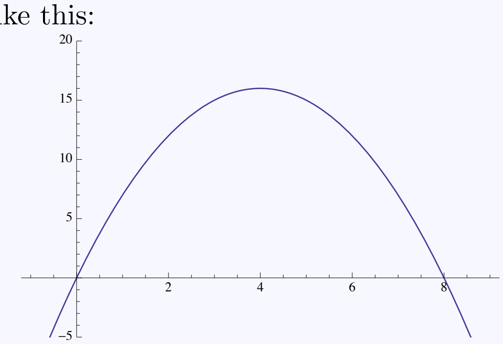
Exercises.

Exercise 1.1. Suppose Paul goes for a bike ride. The distance between Paul and his house is given in miles by the function $d(t) = t(8 - t)$; here t is measured in hours with $t = 0$ corresponding to 7am.

(1) Make a plot of the function d. What domain corresponds to the "physically reasonable" part of the function?
(2) Make a plot of Paul's velocity during the bike ride; you may restrict attention to the "physically reasonable domain." In what units is this velocity measured.
(3) How far does Paul travel during his bike ride?
(4) At what time(s) is Paul the furthest from his house?
(5) At what time(s) is Paul traveling the fastest? At what time is he traveling the slowest? What is his speed at these times?

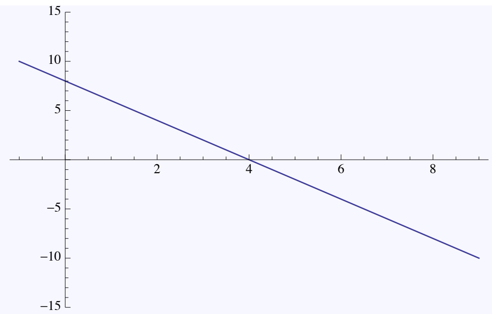
Solution:

(1) The plot looks like this:



The "physically reasonable" part of the domain is $[0, 8]$.

(2) The velocity is $v(t) = 8 - 2t$, the plot of which is this:



The units of velocity are miles per hour.

(3) Paul is furthest from his house at time $t = 4$, when he is 16 miles from home. Thus Paul travels a distance of 32 miles.

(4) Paul is traveling the fastest at $t = 0$ and $t = 8$, when he is traveling 8 miles per hour. He travels slowest at $t = 4$, when his velocity is zero.

Exercise 1.2. Diego Simeone is pacing back and forth along the touchline at the Vicente Calderón stadium. We model the distance between Simeone and his seat by the function $P(t) = 10\cos\left(\frac{\pi}{10}t\right)$, where P is measured in meters and t in minutes; $t = 0$ corresponds to the beginning of the game.

(1) After 45 minutes, when the first half is over, how far has Simeone walked?
(2) At the 45th minute, how far away from his seat is Simeone?
(3) At what club is Simone the manager?

Exercise 1.3. Let's describe Paul's coffee cup as a perfect cylinder with radius 4 cm and height 8 cm. Suppose we measure the height of the coffee in the cup at various times during class and obtain the data as in Table 1.1.

(1) Compute the volume (in cubic centimeters, abbreviated 'cc') of coffee consumed during each time interval.
    Then compute, at each time listed in the left column of Table 1.1, the total coffee consumed up to that point; call this quantity the "cumulative consumption." Make a plot of cumulative consumption vs. time.
(2) Estimate the rate (in cc/min) of coffee consumption during each time interval. Make a plot of consumption rate vs. time.
(3) Explain the relation between the cumulative consumption and the rate of consumption functions. How does this relationship manifest itself in your plots?
    (If you were only given one of the plots, could you reconstruct the other? If so, how? If not, what information is missing?)

Exercise 1.4. Read §1.2 "Calculus without limits" of Strang's book.

(1) Explain in your own words Strang's comments on
    - "Central idea"
    - "Subscript notation"
    - "Examples"

Time elapsed (min)	Height of coffee (cm)
0	7.0
10	5.5
20	5.0
30	3.0
40	1.0
50	0.5

TABLE 1.1. The height of coffee in Paul's mug at various times during one class period.

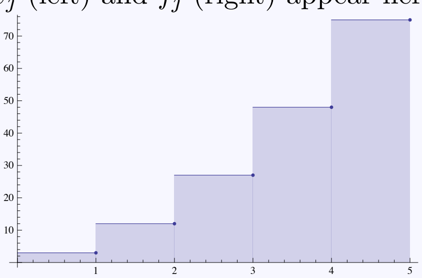
- "Functions"
- "Units"

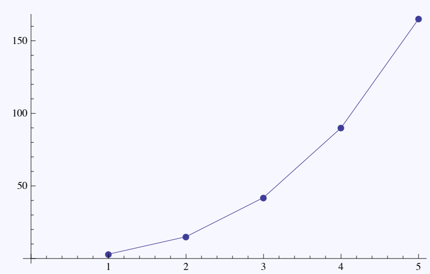
(2) Suppose the values of v are given by $v_j = 3j^2$, for $j = 0, 1, 2, 3, 4, 5$. Supposing that $f_0 = 0$, what are the values of the other f_j? (If you can, find a formula in terms of j!) Make plots of v_j and f_j as done in Strang.

Solution:
The value of f_j is the accumulation of all the preceding v_j's. Thus we have $f_1 = f_0 + v_1$, $f_2 = f_0 + v_1 + v_2$, etc. This leads to the following table.

j	$v_j = 3j^2$	f_j
0	$v_0 = 0$	$f_0 = 0$
1	$v_1 = 3(1)^2 = 3$	$f_1 = 0 + 3 = 3$
2	$v_2 = 3(2)^2 = 12$	$f_2 = 0 + 3 + 12 = 15$
3	$v_3 = 3(3)^3 = 27$	$f_3 = 0 + 3 + 12 + 27 = 42$
4	$v_4 = 3(4)^2 = 48$	$f_4 = 0 + 3 + 12 + 27 + 48 = 90$
5	$v_5 = 3(5)^2 = 75$	$f_5 = 0 + 3 + 12 + 27 + 48 + 75 = 165$

Plots of v_j (left) and f_j (right) appear here:





Bonus: If you remember the formula $1^2 + 2^2 + \cdots + j^2 = \frac{j(j+1)(2j+1)}{6}$ then you can deduce that $f_j = \frac{j(j+1)(2j+1)}{2}$. If you don't remember this formula, or never learned it, do not fear – we'll learn it soon!

(3) Suppose the values of v are given by $v_j = \frac{1}{j^2}$, for $j = 1, 2, 3, 4, 5, 6$. Supposing that $f_1 = 0$, what are the values of the other f_j? (If you can, find a formula in terms of j!) Make plots of v_j and f_j as done in Strang.