$\boxed{15.1}$

$$\mu = \int_0^\infty x\,p(x)\,dx = \lim_{L\to\infty} \int_0^L x e^{-x}\,dx$$

Focus on A.D:

$$\int x e^{-x}\,dx \qquad \begin{array}{ll} f = x & dg = e^{-x}dx \\ df = dx & g = -e^{-x} \end{array}$$

$$= -xe^{-x} - \int -e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x}$$

$$\mu = \lim_{L\to\infty} \left[-xe^{-x} - e^{-x} \right|_0^L$$

$$= \lim_{L\to\infty} \left[\left(-Le^{-L} - e^{-L}\right) - (0 - 1) \right]$$

$$= \lim_{L\to\infty} \left[-\underbrace{\frac{L}{e^{-L}}}_{l'H} - e^{-L} + 1 \right] = \underline{1}$$

15.1 cont

$$\sigma^2 = \lim_{L\to\infty} \int_0^L (x-1)^2 e^{-x}\,dx$$

Focus on AD

$$\int (x-1)^2 e^{-x}\,dx \qquad f = (x-1)^2 \quad dg = e^{-x}dx$$

$$df = 2(x-1)dx \quad g = -e^{-x}$$

$$= -(x-1)^2 e^{-x} + \int 2(x-1)e^{-x}\,dx$$

$$f = 2(x-1) \quad dg = e^{-x}dx$$

$$df = 2dx \quad g = -e^{-x}$$

$$= -(x-1)^2 e^{-x} - 2(x-1)e^{-x} + \int 2e^{-x}\,dx$$

$$= -(x-1)^2 e^{-x} - 2(x-1)e^{-x} - 2e^{-x}$$

$$\sigma^2 = \lim_{L\to\infty} \Big[-(x-1)^2 e^{-x} - 2(x-1)e^{-x} - 2e^{-x}\Big]_0^L$$

$$\sigma^2 = \lim_{L\to\infty} \Big[-e^{-L}\{(L-1)^2 + 2(L-1) + 2\}$$

$$+ e^{0}\{(0-1)^2 + 2(0-1) + 2\}\Big]$$

$$= 1\{1 - 2 + 2\} = 1.$$

Thus $\mu = 1, \quad \sigma = 1.$

$\boxed{15.2}$

$$\sigma^2 = \sum_k (k-\mu)^2 p_k$$

$$= \sum_k (k^2 - 2\mu k + \mu^2) p_k$$

$$= \sum_k k^2 p_k - 2\mu \underbrace{\sum_k k p_k}_{\mu} + \mu^2 \underbrace{\sum_k p_k}_{1}$$

$$= \sum_k k^2 p_k - \mu^2.$$

$\boxed{15.2 \text{ cont}}$

For pdf we have

$$\sigma^2 = \int (x-\mu)^2 p(x)\,dx$$

$$= \int x^2 p(x)\,dx - 2\mu \underbrace{\int x\,p(x)\,dx}_{\mu} + \mu^2 \underbrace{\int p(x)\,dx}_{1}$$

$$= \int x^2 p(x)\,dx - \mu^2 .$$

$\boxed{15.3}$

$$\mu = \sum_{k=1}^{\infty} k\, 2\left(\frac{1}{3}\right)^k = 2\sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^k$$

$$\text{let } f(x) = \sum_{k=1}^{\infty} k x^k = x\sum_{k=1}^{\infty} k x^{k-1}$$

$$= x\sum_{k=1}^{\infty} \frac{d}{dx}\left[x^k\right] = x\frac{d}{dx}\left[\sum_{k=1}^{\infty} x^k\right]$$

$$= x\frac{d}{dx}\left[\sum_{k=0}^{\infty} x^k\right] = x\frac{d}{dx}\left[\frac{1}{1-x}\right]$$

$$= x\frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$$

$$\mu = 2f\left(\frac{1}{3}\right) = 2\frac{\left(\frac{1}{3}\right)}{\left(1-\frac{1}{3}\right)^2} = \frac{\frac{2}{3}}{\left(\frac{2}{3}\right)^2} = \frac{3}{2}.$$

$$\sigma^2 = \sum_{k=1}^{\infty} k^2\, 2\left(\frac{1}{3}\right)^k - \left(\frac{3}{2}\right)^2$$

$$= 2\sum_{k=1}^{\infty} k^2\left(\frac{1}{3}\right)^k - \left(\frac{3}{2}\right)^2$$

$\boxed{15.3 \text{ cont}}$

let $g(x) = \sum_{k=1}^{\infty} k^2 x^k$

$$= x \sum_{k=1}^{\infty} k\, k x^{k-1} = x \sum_{k=1}^{\infty} k \frac{d}{dx}\left[x^k\right]$$

$$= x \frac{d}{dx}\left[\sum_{k=1}^{\infty} k x^k\right] = x \frac{d}{dx}\left[f(x)\right]$$

$$= x \frac{d}{dx}\left[x(1-x)^{-2}\right]$$

$$= x\left[(1-x)^{-2} + x(-2)(1-x)^{-3}(-1)\right]$$

$$= \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}$$

$$\sigma^2 = 2g\left(\tfrac{1}{3}\right) - \left(\tfrac{3}{2}\right)^2 = 2\left[\frac{\frac{1}{3}}{\left(\frac{2}{3}\right)^2} + \frac{2\left(\frac{1}{3}\right)^2}{\left(\frac{2}{3}\right)^3}\right] - \left(\tfrac{3}{2}\right)^2$$

$$= \frac{3}{2} + \frac{3}{2} - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}.$$

$\sigma = \frac{\sqrt{3}}{2}.$

15.4

We have $p_k \geq 0$ so need to check

that $$\sum_{k=0}^{\infty} p_k = 1.$$

But $$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \frac{1}{e^2} \frac{2^k}{k!}$$

$$= \frac{1}{e^2} \sum_{k=0}^{\infty} \frac{1}{k!} 2^k$$

$$= \frac{1}{e^2} e^2 > 1$$

because $$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \quad \text{for any } k!$$

We compute

$$\mu = \sum_{k=0}^{\infty} k p_k = \frac{1}{e^2} \sum_{k=0}^{\infty} k \frac{2^k}{k!} = \frac{1}{e^2} \sum_{k=1}^{\infty} \frac{2^k}{(k-1)!}$$

$$= \frac{1}{e^2} \sum_{l=0}^{\infty} \frac{2^{l+1}}{l!} = 2 \frac{1}{e^2} \sum_{l=0}^{\infty} \frac{2^l}{l!} = 2.$$

$\boxed{13.4 \text{ cont}}$

$$\sigma^2 = \sum_{k=0}^{\infty} k^2 p_k - \mu^2$$

Focus on

$$\sum_{k=0}^{\infty} k^2 p_k = \frac{1}{e^2}\sum_{k=0}^{\infty} k^2 \frac{2^k}{k!}$$

$$= \frac{1}{e^2}\sum_{k=1}^{\infty} \frac{k^2}{k!} 2^k$$

$l = k-1$
$k = l+1$

$$= \frac{1}{e^2}\sum_{k=1}^{\infty} \frac{k}{(k-1)!} 2^k$$

$$= \frac{1}{e^2}\sum_{l=0}^{\infty} \frac{l+1}{l!} 2^{l+1}$$

$$= \frac{1}{e^2}\left(\sum_{l=0}^{\infty} \frac{l}{l!} 2^{l+1} + \sum_{l=0}^{\infty} \frac{1}{l!} 2^{l+1}\right)$$

$$= \underbrace{\frac{1}{e^2}\sum_{l=1}^{\infty} \frac{l}{l!} 2^{l+1}}_{\text{re-index } k=l-1,\ l=k+1} + \underbrace{\frac{2}{e^2}\sum_{l=0}^{\infty} \frac{2^l}{l!}}_{=2 \text{ by earlier work}}$$

$$= \frac{1}{e^2}\sum_{k=0}^{\infty} \frac{1}{k!} 2^{k+2} + 2$$

$\boxed{\text{15.4 cont.}}$

$$= \frac{2^2}{e^2} \underbrace{\sum_{k=0}^{\infty} \frac{2^k}{k!}}_{e^2} + 2$$

$$= 2^2 + 2 = 6$$

Thus $\sigma^2 = 6 - \mu^2 = 6 - 4 = 2$

$\sigma = \sqrt{2}$.