TOPIC 14

Geometric integrals

Exercise 14.1. Here we study the ellipse given by

$$\frac{x^2}{25} + \frac{y^2}{4} = 1.$$

(a) Draw a picture of the ellipse. Where does it intersect the x and y axes?

(b) Construct an integral which represents the area of the ellipse by slicing the ellipse vertically. (Your integral should involve "dx".) Use your integral to find the area of the ellipse.

(c) Construct an integral which represents the area of the ellipse by slicing horizontally. (Your integral should involve "dy.") Evaluate the integral, verifying that you get the same area as before.

(d) Consider now only the right half of the ellipse: that part with $x \geq 0$. What is the area of this region? Find the geometric center of this half of the ellipse.

Exercise 14.2. Here we study the ellipse given by

$$\frac{(x-2)^2}{4} + \frac{(y-5)^2}{9} = 1$$

(a) Draw a picture of the ellipse.

(b) Construct an integral which represents the area of the ellipse by slicing the ellipse vertically. (Your integral should involve "dx".) Use your integral to find the area of the ellipse.

(c) Construct an integral which represents the area of the ellipse by slicing horizontally. (Your integral should involve "dy.") Evaluate the integral, verifying that you get the same area as before.

(d) Use integrals to find the geometric center of the ellipse. [Hint: The answer is $(\bar{x}, \bar{y}) = (2, 5)$.]

Exercise 14.3. Find the area, and center of area, of the region bounded by the curves

$$\frac{x^2}{4} - \frac{y^2}{9} = 1 \quad \text{and} \quad x = 9$$

Exercise 14.4. Here we explore volumes of cones.

(a) Suppose we have a cone which has height of 7 and a circular base with a radius of 5. What is the volume of the cone? Where is the geometric center of the cone?

(b) Now find a general formula for the volume of a cone having height h and a circular base with radius r.

(c) (Optional challenge) Can you find a formula for the volume of a cone with height h, but an *elliptical* base?

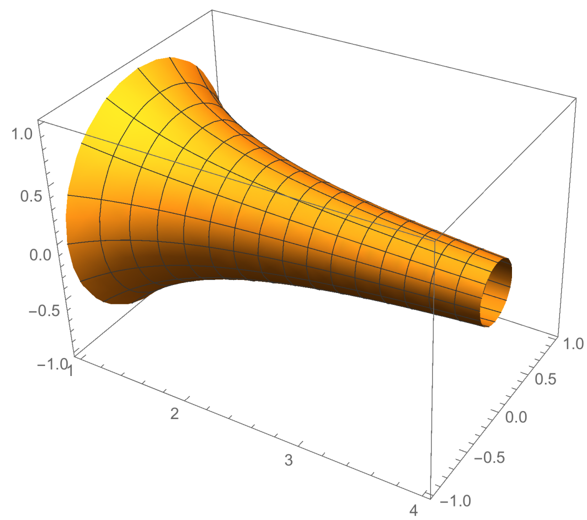
Exercise 14.5. We now repeat the previous problem for square-based pyramids.

(a) Find the volume of a pyramid having height 9 and a square base with sides of length 7. Where is the geometric center of the pyramid?

(b) Find the general formula for a pyramid having height h and a square base with sides of length l.

(c) (Optional challenge) Can you find a formula for a pyramid having height h and a *rectangular* base with sides l and w?

Exercise 14.6. Here we consider various "horns." A typical picture is the following:



(a) Consider the horn described by the function $f(x) = \frac{1}{x}$ for $1 \leq x \leq 10$. Find the volume and center of this horn.

(b) Consider the horn described by the function $f(x) = \frac{1}{x^2}$ for $1 \leq x \leq 10$. Find the volume and center of this horn.

(c) Consider the horn described by the function $f(x) = \frac{1}{x^3}$ for $1 \leq x \leq 10$. Find the volume and center of this horn.

(d) Continue to consider the horn described by $f(x) = \frac{1}{x^3}$, but now for $1 \leq x \leq L$. What happens in the limit as $L \to \infty$? Be sure to examine both the volume and the geometric center.

(e) Repeat the previous part (letting $L \to \infty$) for the earlier horns given by $\frac{1}{x}$ and $\frac{1}{x^2}$. Comment on your findings.