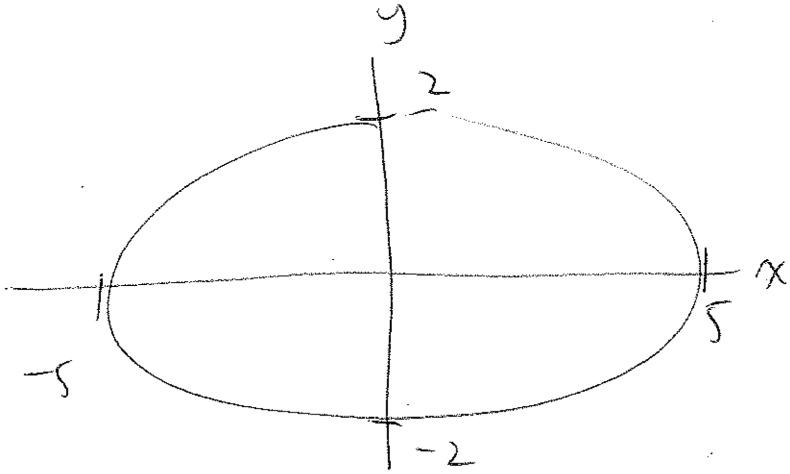
$\boxed{14.1}$

(a)



(b) Solve for $y = \pm 2\sqrt{1 - \frac{x^2}{25}}$

Height of slice is $+2\sqrt{} - -2\sqrt{}$

So

$$dA = 4\sqrt{1 - \frac{x^2}{25}}\,dx$$

$$A = \int_{-5}^{5} 4\sqrt{1 - \frac{x^2}{25}}\,dx \qquad \text{let } v = \frac{x}{5}, \quad dv = \frac{1}{5}dx$$

$$= \int_{-1}^{1} 4\sqrt{1-v^2}\,5\,dv$$

$$= 10 \underbrace{\int_{-1}^{1} 2\sqrt{1-v^2}\,dv}_{= \pi \text{ by earlier HW}}$$

$\boxed{14.1 \text{ cont}}$

So $\quad A = 10\pi$

(c) For horizontal slice we solve for x:

$$x = \pm 5\sqrt{1-\frac{y^2}{4}}$$

$$dA = |x_{right} - x_{left}|\,dy$$

$$= 10\sqrt{1-\frac{y^2}{4}}\,dy$$

$$A = \int_{-2}^{2} 10\sqrt{1-\frac{y^2}{4}}\,dy \qquad w = \frac{y}{2}, \quad dw = \frac{1}{2}dy$$

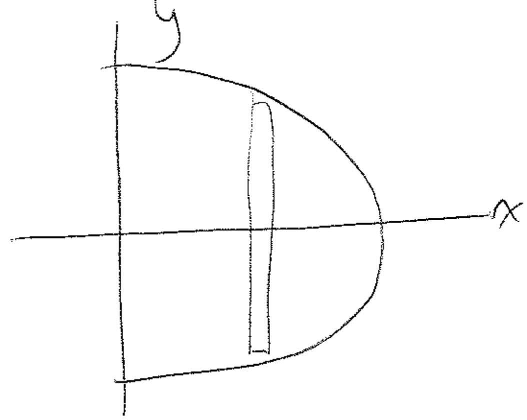
$$= \int_{-1}^{1} 10\sqrt{1-w^2}\,2\,dw$$

$$= 10\underbrace{\int_{-1}^{1} 2\sqrt{1-w^2}\,dw}_{=\pi \text{ by earlier HW}}$$

$$= 10\pi$$

14.1 cont

(d)



Area of half $= 5\pi$

So $\quad x\,\dfrac{dA}{A} = \dfrac{x\;4\sqrt{1-\frac{x^2}{25}}}{5\pi}\,dx$

$$\bar{x} = \int_0^5 x\,\frac{dA}{A} = \int_0^5 \frac{4}{5\pi}\,x\sqrt{1-\frac{x^2}{25}}\,dx$$

let $w = 1 - \dfrac{x^2}{\cancel{3}25}$

$dw = -\dfrac{2}{25}\,x\,dx$

$x\,dx = -\dfrac{25}{2}\,dw$

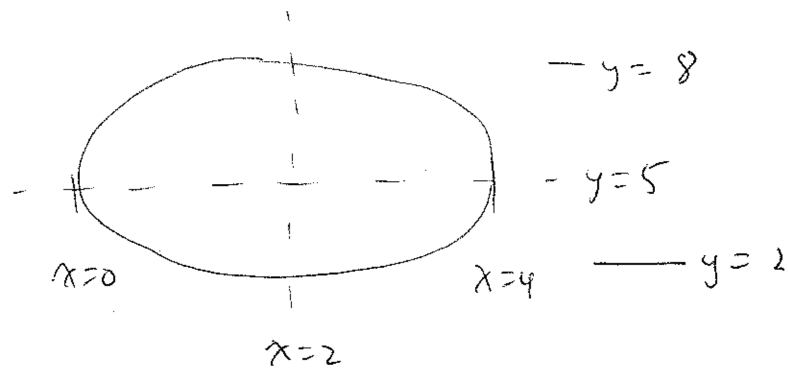
$$\bar{x} = \int_1^0 \frac{4}{5\pi}\sqrt{w}\left(\frac{-25}{2}\right)dw$$

14.1 cont

$$\bar{x} = \frac{-10}{\pi}\frac{2}{3} w^{3/2}\Big|_1^0 = \frac{20}{3\pi}.$$

14.2

(a)



(b) For vertical slice we solve for y:

$$(y-5)^2 = 9\left(1 - \frac{(x-2)^2}{4}\right)$$

$$y = 5 \pm 3\sqrt{1 - \frac{(x-2)^2}{4}}$$

$$dA = \left[y_{top} - y_{bottom}\right] dx$$

$$= \left[(5 + 3\sqrt{}) - (5 - 3\sqrt{})\right] dx$$

$$= 6\sqrt{1 - \frac{(x-2)^2}{4}}\, dx$$

$\boxed{14.2 \text{ cont}}$

$$A = \int_0^4 6\sqrt{1-\left(\frac{x-2}{2}\right)^2}\,dx$$

$$u = \frac{x-2}{2} \qquad du = \frac{1}{2}dx$$

$$A = \int_{-1}^{1} 6\sqrt{1-u^2}\;2\,du$$

$$= 6\int_{-1}^{1} 2\sqrt{1-u^2}\,du = 6\pi$$

(c) For horizontal slicing we solve for x:

$$(x-2)^2 = 4\left(1-\left(\frac{y-5}{3}\right)^2\right)$$

$$x = 2 \pm 2\sqrt{1-\left(\frac{y-5}{3}\right)^2}$$

$$dA = [x_{right} - x_{left}]\,dy$$

$$= [(2+2\sqrt{\ }) - (2-2\sqrt{\ })]\,dy$$

$$= 4\sqrt{1-\left(\frac{y-5}{3}\right)^2}\,dy$$

$\boxed{14.2 \text{ cont}}$

$$A = \int_2^8 4\sqrt{1-\left(\frac{y-5}{3}\right)^2}\,dy \qquad q = \frac{y-5}{3}$$

$$dq = \frac{1}{3}dy$$

$$= \int_{-1}^{1} 4\sqrt{1-q^2}\;3dq$$

$$= 6\int_{-1}^{1}\sqrt{1-q^2}\,dq \quad = 6\pi.$$

(d)
$$\bar{x} = \int_0^4 x\,\frac{6\sqrt{1-\left(\frac{x-2}{2}\right)^2}}{6\pi}\,dx \qquad u = \frac{x-2}{2}$$

$$2u = x-2$$

$$x = 2u+2$$

$$= \int_{-1}^{1}(2u+2)\frac{1}{\pi}\sqrt{1-u^2}\;2du$$

$$= \frac{4}{\pi}\int_{-1}^{1}(u+1)\sqrt{1-u^2}\,du$$

$$= \underbrace{\frac{4}{\pi}\int_{-1}^{1}u\sqrt{1-u^2}\,du}_{=0 \text{ by even/odd}} + \underbrace{\frac{4}{\pi}\int_{-1}^{1}\sqrt{1-u^2}\,du}_{=2 \text{ by earlier work}}$$

$\boxed{14.2 \text{ cont}}$

Alternately

$$\int_{-1}^{1} u\sqrt{1-u^2}\,du \qquad\qquad w = 1-u^2, \quad dw = -2u\,du$$

$$= \int_0^0 \sqrt{w}\left(-\tfrac{1}{2}\right)dw$$

$$= 0 \checkmark$$

Thus $\bar{x} = 2$.

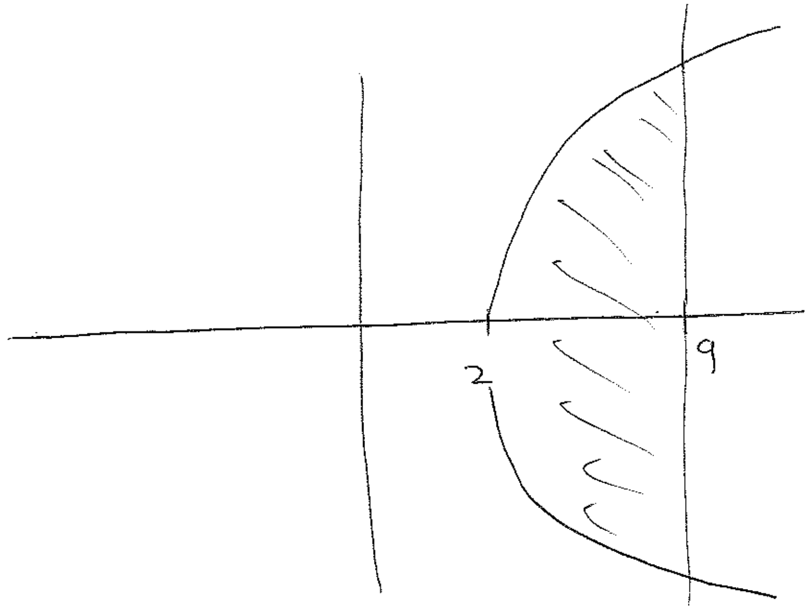
$$\bar{y} = \int_2^8 y \frac{4\sqrt{1-\left(\frac{y-5}{3}\right)^2}}{6\pi}\,dy \qquad q = \frac{y-5}{3}, \quad 3q = y-5, \quad y = 3q+5$$

$$= \int_{-1}^{1} (3q+5)\frac{2}{3\pi}\sqrt{1-q^2}\;3\,dq$$

$$= \underbrace{\frac{6}{\pi}\int_{-1}^{1} q\sqrt{1-q^2}\,dq}_{=0} + \underbrace{\frac{5}{\pi}\int_{-1}^{1} 2\sqrt{1-q^2}\,dq}_{=5}$$

$$= 5$$

$\boxed{14.3}$



If we slice vertically

then we solve for y:

$$\frac{y^2}{9} = \frac{x^2}{4} - 1$$

$$y = \pm 3\sqrt{\frac{x^2}{4} - 1}$$

$$dA = [y_{top} - y_{bottom}]\,dx = 6\sqrt{\left(\frac{x}{2}\right)^2 - 1}\,dx$$

$$A = \int_2^9 6\sqrt{\left(\frac{x}{2}\right)^2 - 1}\,dx \qquad w = \frac{x}{2} \quad dw = \frac{1}{2}dx$$

$$= \int_1^{9/2} 6\sqrt{w^2 - 1}\;2\,dw$$

$\boxed{14.3 \text{ cont}}$

Aside

$$\int \sqrt{w^2-1}\,dw$$

$\underline{\text{IBP!}}$

$f = \sqrt{w^2-1} \quad dg = dw$

$df = \dfrac{w}{\sqrt{w^2-1}}\,dw \quad g = w$

$$= w\sqrt{w^2-1} - \int \frac{w^2}{\sqrt{w^2-1}}\,dw$$

$$= w\sqrt{w^2-1} - \int \frac{w^2-1+1}{\sqrt{w^2-1}}\,dw$$

$$= w\sqrt{w^2-1} - \int \sqrt{w^2-1}\,dw - \ln\left(w+\sqrt{w^2-1}\right)$$

Thus

$$2\int \sqrt{w^2-1}\,dw = w\sqrt{w^2-1} - \ln\left(w+\sqrt{w^2-1}\right)$$

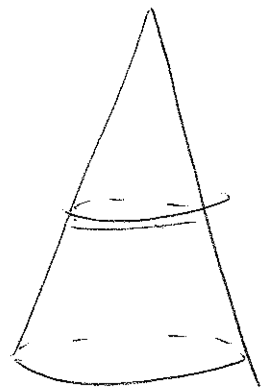
Hence

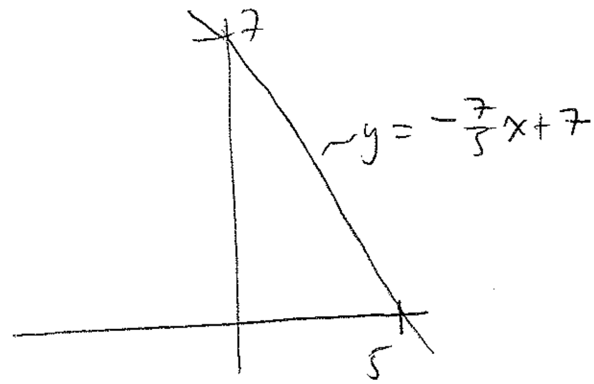
$$A = 3\left[w\sqrt{w^2-1} - \ln\left(w+\sqrt{w^2-1}\right)\right]_1^{9/2}$$

$$= 3\left[\frac{9}{2}\sqrt{\left(\frac{9}{2}\right)^2-1} - \ln\left(\frac{9}{2}+\sqrt{\left(\frac{9}{2}\right)^2-1}\right)\right] - (0).$$

$\boxed{14.4}$

(a)





$\frac{7}{5}x = 7 - y$

$x = 5 - \frac{5}{7}y$

$$dV = \pi\left[5 - \frac{5}{7}y\right]^2 dy$$

$$V = \int_0^7 \pi\left[5 - \frac{5}{7}y\right]^2 dy = \frac{\pi}{3}\left[5 - \frac{5}{7}y\right]^3\left(-\frac{7}{5}\right)\Big|_0^7$$

$$= 0 - -\frac{7}{5}\frac{\pi}{3}[5]^3 = 7(5)^2\frac{\pi}{3}$$

$$\bar{y} = \int_0^7 y\,\frac{dV}{V} = \int_0^7 y\,\frac{\pi\left[5 - \frac{5}{7}y\right]^2}{7(5)^2\frac{\pi}{3}}\,dy$$

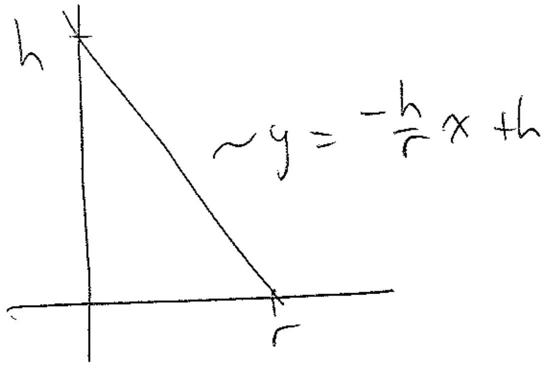
$$= \frac{3}{7\cdot 5^2}\int_0^7 y\left(25 - \frac{10}{7}y + \frac{25}{7^2}y^2\right)dy$$

$$= \frac{3}{7\cdot 5^2}\left[\frac{25}{2}y^2 - \frac{10}{21}y^3 + \frac{25}{7^2\cdot 4}y^4\right|_0^7$$

$$= \frac{3}{7\cdot 5^2}\left[\frac{5^2}{2}7^2 - \frac{5\cdot 2}{3\cdot 7}7^3 + \frac{5^2}{2^2 7^2}7^4\right] = \frac{3}{2}7 - \frac{3\cdot 2\cdot 5\cdot 7}{3} + \frac{3\cdot 7}{4}$$

14.4

(b)



$$\frac{h}{r}x = h - y$$

$$x = r - \frac{r}{h}y$$

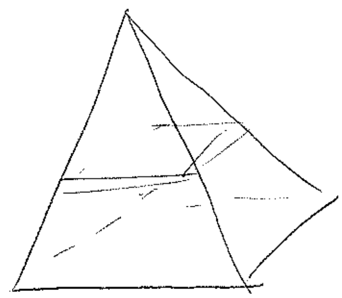
$$dV = \pi\left[r - \frac{r}{h}y\right]^2 dy = \pi r^2\left(1 - \frac{y}{h}\right)^2 dy$$

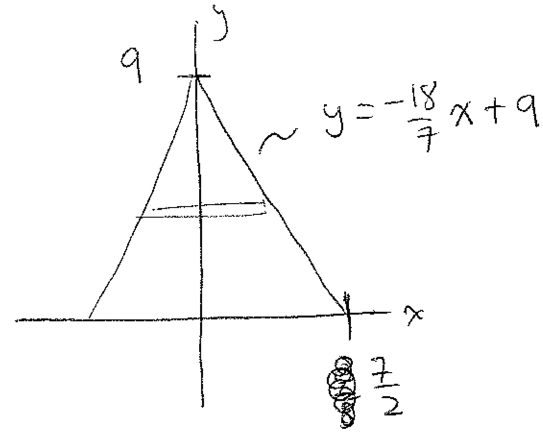
$$V = \int_0^h \pi r^2\left(1 - \frac{y}{h}\right)^2 dy = \frac{\pi r^2 h}{3}\left(1 - \frac{y}{h}\right)^3\Big|_0^h = \frac{\pi r^2 h}{3}.$$

14.5

$\boxed{14.5}$

(a)





Slice horizontally

$$dV = [2x]^2\, dy$$
$$= \left(\frac{7}{9}\right)^2 (9-y)^2\, dy$$

Slope $= -\frac{9}{7/2} = \frac{-18}{7}$

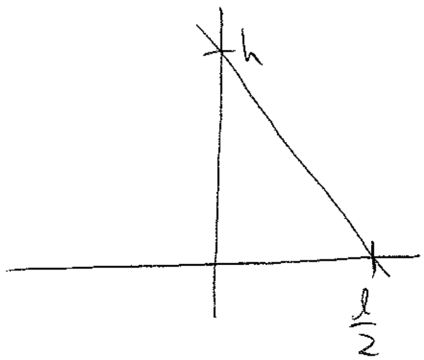
$$\frac{18}{7}x = 9-y$$
$$x = \frac{7}{18}(9-y)$$
$$2x = \frac{7}{9}(9-y) = \left(7 - \frac{7}{9}\right)y$$

$$V = \int_0^9 \left(\frac{7}{9}\right)^2 (9-y)^2\, dy$$
$$= -\frac{7^2}{9^2}\,\frac{1}{3}\,(9-y)^3 \Big|_0^9$$
$$= \frac{7^2}{9^2}\,\frac{1}{3}\,9^3 = \frac{1}{3}\,7^2\,9$$

$$\bar{y} = \int_0^9 y\,\frac{dV}{V} = \frac{3}{7^2 9}\int_0^9 y\left(\frac{7}{9}\right)^2 (9-y)^2\, dy$$
$$= \frac{3}{9^3}\int_0^9 y\,(9^2 - 18y + y^2)\, dy$$
$$= \frac{3}{9^3}\left[\frac{1}{2}9^2(9)^2 - 18\,\frac{1}{3}(9)^3 + \frac{1}{4}(9)^4\right]$$
$$= \frac{3}{2}9 - 18 + \frac{27}{4} = \frac{54 - 72 + 27}{4} = \frac{9}{4}.$$

$\boxed{14.5}$ cont.

(b)



slope $= -\frac{h}{l/2} = \frac{-2h}{l}$

$y = \frac{-2h}{l}x + h$

$= h\left(1 - \frac{2}{l}x\right)$

$\frac{2h}{l}x = h - y$

$x = \frac{l}{2h}(h-y)$

$dV = [2x]^2 dy$

$= \left[\frac{l}{h}(h-y)\right]^2 dy$

$V = \int_0^h \frac{l^2}{h^2}(h-y)^2 dy = -\frac{l^2}{h^2}\frac{(h-y)^3}{3}\Big|_0^h$

$= 0 - \frac{-l^2}{h^2}\frac{h^3}{3} = \frac{1}{3}l^2 h$