TOPIC 13

Integration: Trigonometric functions & their friends

Exercise 13.1. Make a table listing the important properties of the cosine and sine functions. You should include:

(a) the Taylor series for the functions,
(b) a sketch of the graph of the functions,
(c) the Pythogorean identity, together with its geometric interpretation,
(d) the "angle-sum" identities
(e) the "double-angle" formulas, written in a way which is useful for calculus
(f) the derivatives and anti-derivatives of the functions
(g) a sketch of the graph of the inverse functions (What are the domains?),
(h) the derivatives of the inverse functions.

Exercise 13.2. The goal of this problem is to make a analogous table of properties for the hyperbolic cosine and hyperbolic sine functions.

(a) State the definition of the functions in terms of the exponential function.
(b) Use the Taylor series for the exponential function to arrive at the Taylor series for the hyperbolic trigonometric functions.
(c) Sketch the graph of the functions.
(d) Show that the hyperbolic trig funnctions satisfy the the "hyperbolic Pythogorean identity" and give its geometric interpretation.
(e) Show that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.
Look up other such identities on Wikipedia (or your favorite resource). Notice that all the "usual" identities have hyperbolic versions. . . including the "double-angle" formulas, which are useful for calculus!
(f) Notice (from the graphs) that hyperbolic sine has an inverse and, if we restrict to positive numbers, so does hyperbolic cosine. Show that

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad \text{for } x \geq 1,$$
$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}).$$

Explain why the domain of the inverse hyperbolic cosine function is what it is. (Do these functions look familiar?)
(g) Define the hyperbolic tangent function by $\tanh x = \frac{\sinh x}{\cosh x}$. Draw a plot of this function.

Show that $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and that

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$

(Again, does this look familiar?)

(h) What are the derivatives of the inverse hyperbolic cosine, sine, and tangent functions?

(i) Write the following anti-derivatives two different ways (one using logarithms, another using hyperbolic trigonometric functions):

$$\int \frac{1}{\sqrt{x^2-1}}\,dx \qquad \int \frac{1}{\sqrt{x^2+1}}\,dx \qquad \int \frac{1}{x^2-1}\,dx$$

Exercise 13.3. (Calculus with trigonometric functions)

(a) Compute $\int \sin x\,dx$, $\int \sin^2 x\,dx$, $\int \sin^3 x\,dx$, and $\int \sin^4 x\,dx$.

(b) Compute $\int \tan x\,dx$ by writing $\tan x = \frac{1}{\cos x} \sin x$ and changing variables.

(c) Compute $\int \tanh x\,dx$

(d) Show that

$$\frac{1}{\sin x} = \frac{\sin x}{1-\cos^2 x}.$$

Use this to find

$$\int \frac{1}{\sin x}\,dx.$$

(e) Find $\int \frac{1}{\cosh x}\,dx$.

(f) "Recall" the derivatives of the tangent and cotangent functions. Use to find

$$\int \frac{1}{\sin^2 x}\,dx \quad \text{and} \quad \int \frac{1}{\cos^2 x}\,dx.$$

What are the hyperbolic versions of these?

(g) Show that

$$\frac{1}{1+\sin x} = \frac{1-\sin x}{1-\sin^2 x} = \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}.$$

Use this to compute

$$\int \frac{1}{1+\sin x}\,dx.$$

Then compute

$$\int \frac{1}{1-\cos x}\,dx.$$

Exercise 13.4. (Practice with quadratics) Find the following anti-derivatives:

(a) $\int \frac{1}{2x^2+12x+3}\,dx$

(b) $\int \frac{1}{x^2+12x+40}\,dx$

(c) $\int \frac{1}{x^2+12x+36}\,dx$

(d) $\int \frac{1}{\sqrt{2x^2+12x+3}}\,dx$

(e) $\int \frac{1}{\sqrt{x^2+12x+40}}\,dx$

(f) $\int \frac{1}{\sqrt{x^2+12x+36}}\,dx$