TOPIC 12

Integration Techniques II

Exercise 12.1. Complete the following table of "basic anti-derivatives"

Solution:

- $\displaystyle\int \frac{1}{v^2+1}\,dv = \tan^{-1} v + C$
- $\displaystyle\int \frac{1}{v^2-1}\,dv = \frac{1}{2}\ln\left|\frac{v-1}{v+1}\right| + C = -\tanh^{-1} v + C$
- $\displaystyle\int \frac{1}{v^2}\,dv = -\frac{1}{v} + C$
- $\displaystyle\int \frac{1}{\sqrt{v^2+1}}\,dv = \ln\left|v + \sqrt{v^2+1}\right| + C$
- $\displaystyle\int \frac{1}{\sqrt{v^2-1}}\,dv = \ln\left|v + \sqrt{v^2-1}\right| + C$
- $\displaystyle\int \frac{1}{\sqrt{1-v^2}}\,dv = \sin^{-1} v + C$

Exercise 12.2. Compute the following anti-derivatives

(a) $\displaystyle\int \frac{1}{4x^2+24x+32}\,dx$ (b) $\displaystyle\int \frac{1}{4x^2+24x+36}\,dx$ (c) $\displaystyle\int \frac{1}{4x^2+24x+40}\,dx$

Solution:

(a) First, we factor $4x^2+24x+32 = 4\left[(x+3)^2-1\right]$; thus

$$\begin{aligned}\int \frac{1}{4x^2+24x+32}\,dx &= \frac{1}{4}\int \frac{1}{(x+3)^2-1}\,dx \\ &= \frac{1}{4}\,\frac{1}{2}\ln\left|\frac{(x+3)-1}{(x+3)+1}\right| + C \\ &= \frac{1}{8}\ln\left|\frac{x+2}{x+4}\right| + C\end{aligned}$$

(b) We factor $4x^2+24x+36 = 4(x+3)^2$. Thus

$$\begin{aligned}\int \frac{1}{4x^2+24x+36}\,dx &= \int \frac{1}{4(x+3)^2}\,dx \\ &= -\frac{1}{4(x+3)} + C\end{aligned}$$

(c) First, we factor $4x^2 + 24x + 32 = 4\left[(x+3)^2 + 1\right]$; thus

$$\int \frac{1}{4x^2 + 24x + 32}\,dx = \frac{1}{4}\int \frac{1}{(x+3)^2+1}\,dx$$
$$= \frac{1}{4}\tan^{-1}(x+3) + C$$

Exercise 12.3. Compute the following anti-derivatives

(a) $\int \cos\left(\sqrt{x}\right)dx$ (b) $\int \sin(\ln x)\,dx$ (c) $\int \tan^{-1} x\,dx$

Solution:

(a)

$$\int \cos\left(\sqrt{x}\right)dx$$

COV: $u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}\,dx$
$dx = 2u\,du$

$$= \int \cos(u)\,2u\,du$$

IBP:
$f = 2u,\ dg = \cos u\,du$
$df = 2du,\ g = \sin u$

$$= 2u\sin u - \int \sin u\,2du$$
$$= 2u\sin u + 2\cos u$$
$$= 2\sqrt{x}\,\sin\left(\sqrt{x}\right) + 2\cos\sqrt{x} + C$$

(b)

$$\int \sin(\ln x)\,dx$$

$u = \ln x$
$du = \frac{1}{x}dx$
$dx = x\,du = e^u du$

$$= \int \sin(u)e^u\,du$$

Use IBP and wrap-around

$$= \frac{1}{2}e^u(\sin(u) - \cos(u))$$
$$= \frac{1}{2}e^u(\sin(\ln x) - \cos(\ln x)) + C$$

(c)

$$\int \tan^{-1} x \, dx \qquad \left| \begin{array}{l} f = \tan^{-1} x, \ dg = dx \\ df = \frac{1}{1+x^2}\, dx, \ g = x \end{array} \right.$$

$$= x \tan^{-1} x - \int \frac{x}{1+x^2}\, dx \qquad \left| \begin{array}{l} w = 1 + x^2 \\ dw = 2x\, dx \end{array} \right.$$

$$= x \tan^{-1} x - \int \frac{1}{w}\frac{1}{2} dw$$

$$= x \tan^{-1} x - \frac{1}{2} \ln |w| + C$$

$$= x \tan^{-1} x - \frac{1}{2} \ln (1 + x^2) + C$$

Exercise 12.4.

(a) Show that the area of a circle of radius 3 is given by $A = 2\int_{-3}^{3} \sqrt{9 - x^2}\, dx$.

(b) Re-write using $\sqrt{9 - x^2} = 3\sqrt{1 - \left(\frac{x}{3}\right)^2}$.

(c) Introduce a change of variables to reduce to an integral $A = (\#) \int_{?}^{?} \sqrt{1 - v^2}\, dv$.

(d) Use the method of the previous homework to find the anti-derivative: Integration by parts, then add/subtract, and then wrap-around. Conclude $A = 9\pi$.

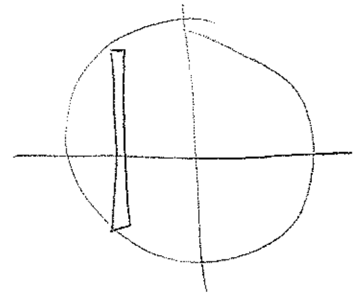
(e) Show that the area of a circle of radius r is πr^2.

Exercise 12.5. Show that the same anti-derivative strategy used for the circle (integration by parts, followed by add/subtract, and then wrap-around) also works for the anti-derivatives

- $\int \sqrt{x^2 + 1}\, dx$
- $\int \sqrt{x^2 - 1}\, dx$
- $\int \sqrt{x^2 + 9}\, dx$

12.4

(a) Circle given by $x^2 + y^2 = 3^2$



Thus $y = \pm\sqrt{9-x^2}$

$y = +\sqrt{9-x^2}$ is top half

$y = -\sqrt{9-x^2}$ is lower half

Slicing vertically

$$dA = \left[\sqrt{9-x^2} - -\sqrt{9-x^2}\right) dx$$

$$= 2\sqrt{9-x^2}\, dx$$

Thus

$$A = \int_{-3}^{3} 2\sqrt{9-x^2}\, dx$$

(b)

$$A = \int_{-3}^{3} 2\sqrt{9-x^2}\, dx = 6\int_{-3}^{3} \sqrt{1-\left(\frac{x}{3}\right)^2}\, dx$$

(c) Let $v = \left(\frac{x}{3}\right)$ so that $dv = \frac{1}{3}\,dx$, $3dv = dx$

$x = -3 \longleftrightarrow v = -1$

$x = 3 \longleftrightarrow v = 1$

$\boxed{12.4, \text{cont.}}$

and

$$A = 6\int_{-1}^{1} \sqrt{1-v^2}\, 3\,dv = 18\int_{-1}^{1} \sqrt{1-v^2}\,dv.$$

(d) We focus on

$$\int \sqrt{1-v^2}\,dv \qquad f = \sqrt{1-v^2} \quad dg = dv$$

$$df = \frac{-v}{\sqrt{1-v^2}}\,dv \qquad g = v$$

$$= v\sqrt{1-v^2} - \int \frac{-v^2}{\sqrt{1-v^2}}\,dv$$

$$= v\sqrt{1-v^2} - \int \frac{1-v^2 \cancel{+} 1}{\sqrt{1-v^2}}\,dv$$

$$= v\sqrt{1-v^2} - \int \frac{1-v^2}{\sqrt{1-v^2}}\,dv + \int \frac{1}{\sqrt{1-v^2}}\,dv$$

$$= v\sqrt{1-v^2} - \int \sqrt{1-v^2}\,dv + \sin^{-1}(v)$$

Thus

$$\int \sqrt{1-v^2}\,dv = \frac{1}{2}\left[v\sqrt{1-v^2} + \sin^{-1}(v)\right] + C$$

$\boxed{\text{D,4 cont}}$

(d) cont.

$$A = 18 \int_{-1}^{1} \sqrt{1-v^2}\, dv$$

$$= 9 \left[v\sqrt{1-v^2} + \sin^{-1}(v) \right]_{-1}^{1}$$

$$= 9 \left[\left(0 + \frac{\pi}{2}\right) - \left(0 - \frac{\pi}{2}\right) \right] = 9\pi$$

(e) For a circle of radius r we have

$$y = \pm \sqrt{r^2 - x^2}$$

Thus

$$A = \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx = 2r \int_{-r}^{r} \sqrt{1 - \left(\frac{x}{r}\right)^2}\, dx$$

Let $v = \frac{x}{r}$ $\quad dv = \frac{1}{r} dx$

$$A = 2r^2 \int_{-1}^{1} \sqrt{1-v^2}\, dv$$

$$= r^2 \left[v\sqrt{1-v^2} + \sin^{-1}(v) \right]_{-1}^{1}$$

$$= \pi r^2$$

$\boxed{12.5}$

- $\displaystyle\int \sqrt{x^2+1}\,dx$

$$f = \sqrt{x^2+1} \qquad dg = dx$$
$$df = \frac{x}{\sqrt{x^2+1}}\,dx \qquad g = x$$

$$= x\sqrt{x^2+1} - \int \frac{x^2}{\sqrt{x^2+1}}\,dx$$

$$= x\sqrt{x^2+1} - \int \frac{x^2+1-1}{\sqrt{x^2+1}}\,dx$$

$$= x\sqrt{x^2+1} - \int \frac{x^2+1}{\sqrt{x^2+1}}\,dx + \int \frac{1}{\sqrt{x^2+1}}\,dx$$

$$= x\sqrt{x^2+1} - \int \sqrt{x^2+1}\,dx + \ln\left|x+\sqrt{x^2+1}\right|$$

Thus

$$\int \sqrt{x^2+1}\,dx = \frac{1}{2}\left[x\sqrt{x^2+1} + \ln\left(x+\sqrt{x^2+1}\right)\right].$$

$\boxed{12.5 \text{ cont}}$

- $\int \sqrt{x^2-1}\,dx$

$f = \sqrt{x^2-1} \qquad dg = dx$

$df = \dfrac{x}{\sqrt{x^2-1}}\,dx \qquad g = x$

$$= x\sqrt{x^2-1} - \int \frac{x^2}{\sqrt{x^2-1}}\,dx$$

$$= x\sqrt{x^2-1} - \int \frac{x^2-1+1}{\sqrt{x^2-1}}\,dx$$

$$= x\sqrt{x^2-1} - \int \sqrt{x^2-1}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx$$

$$= x\sqrt{x^2-1} - \int \sqrt{x^2-1}\,dx - \ln\left|x+\sqrt{x^2-1}\right|$$

Thus $\displaystyle\int \sqrt{x^2-1}\,dx = \frac{1}{2}\left[x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right]$

$\boxed{12.5 \text{ cont}}$

- $\int \sqrt{x^2+9}\, dx$

$$= 3\int \sqrt{\left(\frac{x}{3}\right)^2+1}\, dx \qquad v = \frac{x}{3} \qquad dv = \frac{1}{3}dx$$

$$= 9\int \sqrt{v^2+1}\, dv$$

$$= \frac{9}{2}\left[v\sqrt{v^2+1} + \ln\left(v+\sqrt{v^2+1}\right)\right]$$

$$= \frac{9}{2}\left[\frac{x}{3}\sqrt{\left(\frac{x}{3}\right)^2+1} + \ln\left(\frac{x}{3}+\sqrt{\left(\frac{x}{3}\right)^2+1}\right)\right]$$