TOPIC 11

Integration Techniques I

Main ideas.

- Basic anti-derivatives
- Integration by parts
- Application to area and center of area

Exercise 11.1.

(a) Compute the derivative $\frac{d}{dx}\left[\ln\left(x+\sqrt{x^2+1}\right)\right]$ and fully simplify. Use the result to complete the following:

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \boxed{}$$

(b) Compute the derivative $\frac{d}{dx}\left[\ln\left(x+\sqrt{x^2-1}\right)\right]$ and fully simplify. What anti-derivative do you now know?

Solution:

(a) $\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln|x+\sqrt{x^2+1}| \quad +C$

(b) $\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}| \quad +C$

Exercise 11.2. Use partial fractions to write $\dfrac{1}{x^2-1}$ as the sum of two fractions. Use this to compute $\displaystyle\int_{-1/2}^{1/2} \frac{1}{x^2-1}\,dx$.

Solution:

We have

$$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

Thus

$$\begin{aligned}\int_{-1/2}^{1/2} \frac{1}{x^2-1}\,dx &= \left[\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right]_{-1/2}^{1/2} \\ &= \frac{1}{2}\ln\left(\frac{\frac{1}{2}}{\frac{3}{2}}\right) - \frac{1}{2}\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) \\ &= \frac{1}{2}\ln\left(\frac{1}{2}\frac{2}{3}\frac{2}{3}\frac{1}{2}\right) \\ &= -\ln 3\end{aligned}$$

Exercise 11.3. Find the anti-derivatives $\displaystyle\int xe^{2x}\,dx$ and $\int x^2e^{2x}\,dx$.

Solution:

First we compute.

$$\int xe^{2x}\,dx \qquad \left|\quad \begin{array}{l} f = x,\ dg = e^{2x}dx \\ df = dx,\ g = \frac{1}{2}e^{2x}\end{array}\right.$$

$$\begin{aligned} &= x\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} \quad + C\end{aligned}$$

Next we compute

$$\int x^2e^{2x}\,dx \qquad \left|\quad \begin{array}{l} f = x^2,\ dg = e^{2x}\,dx \\ df = 2x\,dx,\ \frac{1}{2}e^{2x}\end{array}\right.$$

$$\begin{aligned} &= \frac{1}{2}x^2e^{2x} - \int \frac{1}{2}e^{2x}2x\,dx \\ &= \frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right) \quad + C \\ &= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} \quad + C\end{aligned}$$

Exercise 11.4. Compute the definite integral $\int_0^\pi x \cos x \, dx$.

Solution:

We compute using integration by parts

$$\int_0^\pi x \cos x \, dx \qquad \left| \begin{array}{l} f = x, \; dg = \cos x \, dx \\ df = dx, \; g = \sin x \end{array} \right.$$

$$\begin{aligned} &= \Big[x \sin x\Big]_0^\pi - \int_0^\pi \sin x \, dx \\ &= 0 + \Big[\cos x\Big]_0^\pi \\ &= -2 \end{aligned}$$

Exercise 11.5. Compute the definite integral $\int_0^\pi e^x \sin x \, dx$

Solution:

First we compute the anti-derivative using integration by parts twice.

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx \qquad \left| \begin{array}{l} f = \sin x, \; dg = e^x \, dx \\ df = \cos x \, dx, \; g = e^x \end{array} \right.$$

$$= e^x \sin x - \left[e^x \cos x - \int -\sin x e^x \, dx\right] \qquad \left| \begin{array}{l} f = \cos x, \; dg = e^x \, dx \\ df = -\sin x \, dx, \; g = e^x \end{array} \right.$$

Hence

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

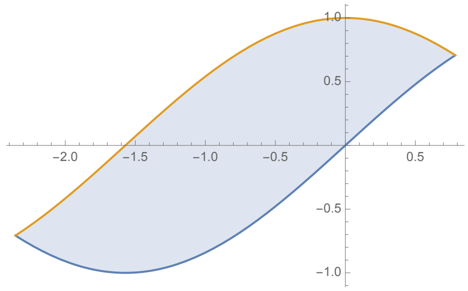
and thus if we solve for the original integral we find

$$\int e^x \sin x \, dx = \frac{1}{2} \left[e^x \sin x - e^x \cos x\right].$$

Now we return to the original problem:

$$\begin{aligned} \int_0^\pi e^x \sin x \, dx &= \frac{1}{2} \Big[e^x \sin x - e^x \cos x\Big]_0^\pi \\ &= \frac{1}{2} \left(e^\pi + 1\right) \end{aligned}$$

Exercise 11.6. Find the area of the region bounded by the graphs of the sine and cosine functions as shown here:



Then find x coordinate of the center of the region.

Solution:

Slicing horizontally, we find that the area of each slice is

$$dA = [\cos x - \sin x]\, dx$$

The first slice is at $x = -\frac{3\pi}{4}$ and the last slice is at $x = \frac{\pi}{4}$ and hence

$$A = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} [\cos x - \sin x]\, dx = 2\sqrt{2}.$$

To find the x coordinate of the center, we compute

$$\bar{x} = \int_{\text{start}}^{\text{end}} x \frac{dA}{A} = \frac{1}{2\sqrt{2}} \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} x[\cos x - \sin x]\, dx = \cdots = -\frac{\pi}{4}$$

Exercise 11.7. Compute $\lim_{n\to\infty} \int_0^n e^{-x}\, dx$. Interpret the result in terms of area. Then find the x coordinate of the center of the region.

Solution:

We can interpret the integral as the area of the following region:

and compute

$$\lim_{n\to\infty} \int_0^n e^{-x}\, dx = \cdots = 1.$$

The center we may compute by

$$\bar{x} = \lim_{n\to\infty} \int_0^n xe^{-x}\,dx = \cdots = 1.$$

Exercise 11.8. In this problem you compute the area of a circle of radius 1.

(a) Show that the area of the circle can be represented by the integral

$$A = 2\int_{-1}^{1} \sqrt{1-x^2}\,dx.$$

(b) Use integration by parts to conclude that

$$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx$$

(c) Algebraically rearrange this in to the form

$$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

and conclude that

$$2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

(d) Return to the definite integral and conclude that the area is π.

Solution:

$$\begin{aligned} 2\int_{-1}^{2} \sqrt{1-x^2}\,dx &= \left[x\sqrt{1-x^2} + \sin^{-1} x\right]_{-1}^{1} \\ &= \left[0 + \frac{\pi}{2}\right] - \left[0 - \frac{\pi}{2}\right] \\ &= \pi \end{aligned}$$