TOPIC 10

Convergence and remainders for Taylor series

PRINCIPLE (Integral formula for Taylor remainder). *For any function f (that can be differentiated as many times as we want) we have*

$$\begin{aligned} f(x) &= p_n(x) + R_n \\ &= a_0 + a_1(x - x_*) + a_2(x - x_*)^2 + \cdots + a_n(x - x_*)^n + R_n, \end{aligned}$$

where

$$a_k = \frac{f^{(k)}(x_*)}{k!} \qquad \text{and} \qquad R_n = \int_{x_*}^{x} \frac{1}{n!}(x - y)^n f^{(n+1)}(y)\, dy.$$

PRINCIPLE (Cauchy's formula for Taylor remainder). *For any function f (that can be differentiated as many times as we want) we have*

$$\begin{aligned} f(x) &= p_n(x) + R_n \\ &= a_0 + a_1(x - x_*) + a_2(x - x_*)^2 + \cdots + a_n(x - x_*)^n + R_n, \end{aligned}$$

where

$$a_k = \frac{f^{(k)}(x_*)}{k!} \qquad \text{and} \qquad R_n = \frac{1}{n!}(x - c)^n (x - x_*) f^{(n+1)}(c),$$

for some c between x and x_*.

PRINCIPLE (Simple estimate for Taylor remainder). *Suppose that*

$$|f^{(n+1)}(y)| \le M \qquad \text{for all } y \text{ between } x \text{ and } x_*$$

Then

$$|R_n| \le \frac{M}{(n+1)!}|x - x_*|^{n+1}.$$

Exercise 10.1. Suppose we are interested in the function e^x for x between 0 and 20. How good of an approximation is the 10^{th} order Taylor polynomial, if the polynomial is centered at $x_* = 0$?

Exercise 10.2. Suppose we want to study the cosine function on the interval $[0, 2\pi]$ and want errors to be less than 10^{-4}. Which order Taylor approximation (centered at $x_* = 0$) is sufficient?