

TOPIC 9

Computing Taylor series

Exercise 9.1. Memorize the following

$$\begin{aligned} \frac{1}{1-x} &\sim \sum_{k=0}^{\infty} 1 = 1 + x + x^2 + x^3 + \dots \\ e^x &\sim \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \\ \cos x &\sim \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \dots \\ \sin x &\sim \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 - \dots \end{aligned}$$

For which values of x do each of the series converge?

Exercise 9.2. Use the basic series above to find series expansions for the following functions. Be sure to indicate for which values of x the series converge.

- | | | |
|---------------------------|---------------------------|----------------------------|
| (1) $\sin(2x)$ | (5) $\frac{\sin(5x)}{2x}$ | (8) $x^2 \cos(3x)$ |
| (2) $e^{-\frac{1}{2}x^2}$ | (6) $\frac{e^x - 1}{x}$ | (9) $\frac{x}{1+x^2}$ |
| (3) $\frac{4}{4-x}$ | (7) $\frac{1}{1+x^2}$ | (10) $\frac{1-x^2}{1+x^2}$ |
| (4) 10^x | | |

[Hint: $10 = e^{\ln 10}$.]

Exercise 9.3. Use calculus to find series expansions for the following functions.

- (1) $\ln(1-x)$
- (2) $\tan^{-1} x$

then find series expansions for the following functions

- (3) $\ln(1+x^2)$
- (4) $\ln(4-x)$
- (5) $\ln\left(\frac{1+x}{1-x}\right)$ [Hint: Logarithm identities.]
- (6) $x \tan^{-1} x$

Exercise 9.4. Use change of variables to find...

- (1) ... a series expansion for e^x centered at $x_* = 2$.
- (2) ... a series expansion for $\cos x$ centered at $x_* = \frac{\pi}{2}$.
- (3) ... a series expansions for $\ln x$ centered at $x_* = 1$.

- (4) ... a series expansions for $\ln x$ centered at $x_* = 4$.

Exercise 9.5. Here we study the function $f(x) = \sqrt{1+x}$ and its friends.

- (1) Find a formula for the Taylor polynomials for f , centered at $x_* = 0$.
- (2) Construct the Taylor series for f . For which values of x does the series converge?
- (3) Use calculus to find a series expansion for the function $(1+x)^{-1/2}$. Where does the series converge?
- (4) Find a series expansions for the function $(1-4x^2)^{1/2}$. Where does the series converge?