TOPIC 9

# Computing Taylor series

**Exercise 9.1.** Memorize the following

$$\frac{1}{1-x} \sim \sum_{k=0}^{\infty} = 1 + x + x^2 + x^3 + \dots$$

$$e^x \sim \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

$$\cos x \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \dots$$

$$\sin x \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} = x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 - \dots$$

For which values of $x$ do each of the series converge?

**Solution:**

The first series converges for $|x| < 1$, while the rest converge for all $x$. This is easily checked using the ratio test. For example, for the second series we have the ratio

$$\frac{\left|\frac{x^{k+1}}{(k+1)!}\right|}{\left|\frac{x^k}{k!}\right|} = \frac{|x|}{k} \to 0$$

as $k \to \infty$ no matter what $x$ is.

**Exercise 9.2.** Use the basic series above to find series expansions for the following functions. Be sure to indicate for which values of $x$ the series converge.

(1) $\sin(2x)$

(2) $e^{-\frac{1}{2}x^2}$

(3) $\dfrac{4}{4-x}$

(4) $10^x$
[Hint: $10 = e^{\ln 10}$.]

(5) $\dfrac{\sin(5x)}{2x}$

(6) $\dfrac{e^x - 1}{x}$

(7) $\dfrac{1}{1+x^2}$

(8) $x^2\cos(3x)$

(9) $\dfrac{x}{1+x^2}$

(10) $\dfrac{1-x^2}{1+x^2}$

**Solution:**

(1) The following converges for all $x$

$$\sin(2x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} x^{2k+1} = 2x - \frac{4}{6}x^3 + \frac{8}{5!}x^5 - \ldots$$

(2) We have convergence for all $x$:

$$e^{-\frac{1}{2}x^2} \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \ldots$$

(3) Re-writing the function we have

$$\frac{4}{4-x} = \frac{1}{1-\frac{x}{4}} \sim \sum_{k=0}^{\infty} \frac{1}{4^k} x^k = 1 + \frac{1}{4}x + \frac{1}{16}x^2 + \ldots$$

which converges for $|x| < 4$.

(4) We rewrite the function and obtain

$$10^x = e^{(\ln 10)x} \sim \sum_{k=0}^{\infty} \frac{(\ln 10)^k}{k!} x^k = 1 + \ln 10\, x + \frac{1}{2}(\ln 10)x^2 + \ldots,$$

which converges for all $x$.

(5) First we write

$$\sin(5x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k 5^{2k+1}}{(2k+1)!} x^{2k+1} = 5x - \frac{5^3}{3!}x^3 + \frac{5^5}{5!}x^5 - \ldots$$

Then we divide by $2x$ to obtain

$$\frac{\sin(5x)}{2x} \sim \sum_{k=0}^{\infty} \frac{(-1)^k 5^{2k}}{2(2k+1)!} x^{2k+1} = \frac{5}{2} - \frac{5^3}{2 \cdot 3!}x^2 + \frac{5^5}{2 \cdot 5!}x^4 - \ldots$$

which converges for all $x$.

(6) Subtracting the first term from the series for $e^x$ yields

$$e^x - 1 \sim \sum_{k=1}^{\infty} \frac{1}{k!} x^k = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$$

Dividing, we obtain

$$\frac{e^x - 1}{x} \sim \sum_{k=1}^{\infty} \frac{1}{k!} x^{k-1} = 1 + \frac{1}{2}x + \frac{1}{6}x^2 + \ldots$$

which converges for all $x$.

(7) Using the geometric series we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \sim \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - \ldots$$

The series converges for $|x| < 1$.

(8) Substitute $3x$ in to the series for cosine, then multiply by $x^2$ to obtain

$$x^2 \cos(3x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k}}{(2k)!} x^{2k+2} = x^2 - \frac{9}{2}x^4 + \dots$$

which converges for all $x$.

(9) Simply multiply the earlier answer by $x$ to obtain

$$\frac{x}{1+x^2} \sim \sum_{k=0}^{\infty} (-1)^k x^{2k+1} = x - x^3 + x^5 - \dots$$

The series converges for $|x| < 1$.

(10) This requires a bit of strategy. First, multiply the earlier answer by $x^2$ to obtain

$$\frac{x^2}{1+x^2} \sim \sum_{k=0}^{\infty} (-1)^k x^{2k+2} = x^2 - x^4 + x^6 - \dots,$$

which converges for $|x| < 1$. Then re-index the sum, setting $l = k + 1$

$$\frac{x^2}{1+x^2} \sim \sum_{l=1}^{\infty} (-1)^{l-1} x^{2l} = x^2 - x^4 + x^6 - \dots.$$

Now we write down the earlier answer with index $l$:

$$\frac{1}{1+x^2} \sim \sum_{l=0}^{\infty} (-1)^l x^{2l} = 1 - x^2 + x^4 - \dots$$

Finally, we subtract the two in order to obtain

$$\begin{aligned}
\frac{1-x^2}{1+x^2} &= \frac{1}{1+x^2} - \frac{x^2}{1+x^2} \\
&= 1 + \sum_{l=1}^{\infty} \left( (-1)^l x^{2l} - (-1)^{l-1} x^{2l} \right) \\
&= 1 + \sum_{l=1}^{\infty} 2(-1)^l x^{2l} \\
&= 1 - 2x^2 + 2x^4 - 2x^6 + \dots
\end{aligned}$$

Since all the intermediate series converge for $|x| < 1$, so does this one.

**Exercise 9.3.** Use calculus to find series expansions for the following functions.

(1) $\ln(1-x)$
(2) $\tan^{-1} x$

then find series expansions for the following functions

(3) $\ln(1+x^2)$
(4) $\ln(4-x)$
(5) $\ln\left(\frac{1+x}{1-x}\right)$ [Hint: Logarithm identities.]
(6) $x\tan^{-1} x$

**Solution:**

(1) Integrating the geometric series we have

$$-\ln(1-x) \sim C + \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} = C + x + \frac{1}{2}x^2 + \dots$$

Evaluating at $x = 0$ shows us that the constant $C$ is zero; multiplying both sides by a minus sign gives

$$\ln(1-x) \sim -\sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} = -x - \frac{1}{2}x^2 + \dots$$

If we want, we can re-index to obtain the simpler formula

$$\ln(1-x) \sim -\sum_{k=1}^{\infty} \frac{1}{k} x^{k} = -x - \frac{1}{2}x^2 + \dots.$$

We now check convergence:

- For $|x| < 1$ the ratio test gives convergence.
- When $x = 1$ the series diverges by $p$-series.
- When $x = -1$ the alternating principle gives convergence.

Thus the series is valid for $-1 \le x < 1$.

(2) We know from the previous exercise that

$$\frac{1}{1+x^2} \sim \sum_{l=0}^{\infty} (-1)^l x^{2l} = 1 - x^2 + x^4 - \dots$$

Integrating both sides gives

$$\tan^{-1} x \sim C + \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} x^{2l+1} = C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$$

Evaluating at $x = 0$ shows us that $C = 0$ and thus

$$\tan^{-1} x \sim \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} x^{2l+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$$

We now check for convergence:

- When $|x| < 1$ the ratio test gives convergence.

- When $x = 1$ the alternating principle gives convergence.
- When $x = -1$ the series diverges by comparison with $p$-series.

Thus the series is valid for $-1 < x \leq 1$.

(3) Using the previous result we have

$$\begin{aligned} \ln(1 + x^2) &\sim -\sum_{k=1}^{\infty} \frac{1}{k}(-x^2)^k \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^{2k} \end{aligned}$$

We furthermore have convergence when $-1 \leq -x^2 < 1$, which occurs when $|x| \leq 1$.

(4) We write

$$\begin{aligned} \ln(4 - x) &= \ln(4(1 - \frac{1}{4}x)) \\ &= \ln 4 + \ln(1 - \frac{1}{4}x) \\ &\sim \ln 4 - \sum_{k=1}^{\infty} \frac{1}{k+1} \frac{1}{4^k} x^{k+1} \\ &= \ln 4 - \frac{1}{4}x - \frac{1}{2}\frac{1}{16}x^2 - \frac{1}{3}\frac{1}{4^3}x^3 - \dots \end{aligned}$$

with convergence for $-4 \leq x < 4$.

**Exercise 9.4.** Use change of variables to find...

(1) ...a series expansion for $e^x$ centered at $x_* = 2$.
(2) ...a series expansion for $\cos x$ centered at $x_* = \frac{\pi}{2}$.
(3) ...a series expansions for $\ln x$ centered at $x_* = 1$.
(4) ...a series expansions for $\ln x$ centered at $x_* = 4$.

**Solution:**

(1) We write

$$\begin{aligned} e^x &= e^{(x-2)+2} \\ &= e^2 \sum_{k=0}^{\infty} \frac{1}{k!}(x-2)^k \\ &= \sum_{k=0}^{\infty} \frac{e^2}{k!}(x-2)^k \end{aligned}$$

This converges for all $x$.

(2) Using trig identities we have

$$\begin{aligned} \cos x &= \cos\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= -\sin\left(x - \frac{\pi}{2}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}\left(x - \frac{\pi}{2}\right)^{2k+1} \end{aligned}$$

This converges for all $x$.

(3) We have

$$\begin{aligned} \ln x = \ln(1 + x - 1) &= \ln(1 - [-(x-1)]) \\ &= -\sum_{k=1}^{\infty} \frac{1}{k}[-(x-1)]^k \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k \end{aligned}$$

Based on work done previously, this converges when $-1 \leq [-(x-1)] < 1$, which is equivalent to $0 < x \leq 2$.

(4) We write

$$\begin{aligned}\ln x &= \ln(4 + x - 4)\\ &= \ln\left(4\left[1 + \frac{x-4}{4}\right]\right)\\ &= \ln 4 + \ln\left(1 + \frac{x-4}{4}\right)\\ &= \ln 4 - \sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{x-4}{4}\right)^k\\ &= \ln 4 + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k\,4^k}(x-4)^k\end{aligned}$$

Based on work done above, this converges for $0 < x \le 8$.

**Exercise 9.5.** Here we study the function $f(x) = \sqrt{1+x}$ and its friends.

(1) Find a formula for the Taylor polynomials for $f$, centered at $x_* = 0$.
(2) Construct the Taylor series for $f$. For which values of $x$ does the series converge?
(3) Use calculus to find a series expansion for the function $(1+x)^{-1/2}$. Where does the series converge?
(4) Find a series expansions for the function $(1-4x^2)^{1/2}$. Where does the series converge?

**Solution:**

(1) We compute a bunch of derivatives and deduce that

$$\begin{aligned}f(0) &= 1\\ f'(0) &= \frac{1}{2}\\ &\vdots\\ f^{(k)}(0) &= \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\cdots\left(\frac{-(2k-3)}{2}\right)\end{aligned}$$

Thus the Taylor coefficients are given by

$$\begin{aligned}a_0 &= 1\\ a_1 &= \frac{1}{2}\\ &\vdots\\ a_k &= \frac{(-1)^{k-1}(1)(3)\dots(2k-3)}{2^k\,k!}\end{aligned}$$

and the Taylor polynomial of order $k$ is

$$p_k(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots + \frac{(-1)^{k-1}(1)(3)\dots(2k-3)}{2^k\,k!}x^k.$$

(2) The Taylor series is thus

$$f(x) \sim 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)(3)\dots(2k-3)}{2^k\, k!}\, x^k.$$

We use the ratio test to investigate convergence:

$$\begin{aligned} \text{ratio} &= \frac{|(-1)^{(k+1)-1}|(1)(3)\dots(2(k+1)-3)|x|^{k+1}}{2^{(k+1)}\,(k+1)!} \div \frac{|(-1)^{k-1}|(1)(3)\dots(2k-3)|x|^k}{2^k\, k!} \\ &= \frac{(2k-1)|x|}{2(k+1)} \\ &\to |x| \qquad \text{as } k \to \infty. \end{aligned}$$

Thus we have absolute convergence when $|x| < 1$. When $x = 1$ the series converges by the alternating principle. When $x = -1$ the series looks like

$$1 - \frac{1}{2} - \frac{1}{2\cdot 4} - \frac{1\cdot 3}{2\cdot 4\cdot 6} - \dots - \frac{1\cdot 3\cdots(2k-3)}{2\cdot 4\cdot 6\cdots(2k)}$$

This actually diverges, but it is not so easy to see why...

(3) Notice that

$$\frac{d}{dx}\left[\sqrt{1+x}\right] = \frac{1}{2}(1+x)^{-1/2}$$

Thus we can simply take the derivative, and multiply by 2, to conclude that

$$(1+x)^{-1/2} \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)(3)\dots(2k-3)}{2^{k-1}\,(k-1)!}\, x^{k-1}$$

If we reindex we obtain

$$(1+x)^{-1/2} \sim \sum_{l=0}^{\infty} \frac{(-1)^{l}(1)(3)\dots(2l-1)}{2^{l}\, l!}\, x^{l}$$

(4) Substituting in $-x^2$ for $x$ yields

$$\begin{aligned} (1-4x^2)^{1/2} &\sim 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)(3)\dots(2k-3)}{2^k\, k!}\, (-x^2)^k \\ &\sim 1 - \sum_{k=1}^{\infty} \frac{(1)(3)\dots(2k-3)}{2^k\, k!}\, x^{2k} \end{aligned}$$

TOPIC 10

# Convergence and remainders for Taylor series

PRINCIPLE (Integral formula for Taylor remainder). *For any function $f$ (that can be differentiated as many times as we want) we have*

$$\begin{aligned} f(x) &= p_n(x) + R_n \\ &= a_0 + a_1(x-x_*) + a_2(x-x_*)^2 + \cdots + a_n(x-x_*)^n + R_n, \end{aligned}$$

*where*

$$a_k = \frac{f^{(k)}(x_*)}{k!} \qquad \text{and} \qquad R_n = \int_{x_*}^{x} \frac{1}{n!}(x-y)^n f^{(n+1)}(y)\,dy.$$

PRINCIPLE (Cauchy's formula for Taylor remainder). *For any function $f$ (that can be differentiated as many times as we want) we have*

$$\begin{aligned} f(x) &= p_n(x) + R_n \\ &= a_0 + a_1(x-x_*) + a_2(x-x_*)^2 + \cdots + a_n(x-x_*)^n + R_n, \end{aligned}$$

*where*

$$a_k = \frac{f^{(k)}(x_*)}{k!} \qquad \text{and} \qquad R_n = \frac{1}{n!}(x-c)^n(x-x_*)f^{(n+1)}(c),$$

*for some $c$ between $x$ and $x_*$.*

PRINCIPLE (Simple estimate for Taylor remainder). *Suppose that*

$$|f^{(n+1)}(y)| \le M \qquad \text{for all } y \text{ between } x \text{ and } x_*$$

*Then*

$$|R_n| \le \frac{M}{(n+1)!}|x-x_*|^{n+1}.$$

**Exercise 10.1.** Suppose we are interested in the function $e^x$ for $x$ between 0 and 20. How good of an approximation is the $10^{\text{th}}$ order Taylor polynomial, if the polynomial is centered at $x_* = 0$?

**Solution:**

Using the simple estimate with $f(x) = e^x$, we have $|f^{(n+1)}(x)| = e^x \leq e^2 0$ on this interval. Thus the error is less than

$$\frac{e^{20} 20^{n+1}}{11!}$$

This is not a small number and it tells us that for such a large interval, we need a much higher order polynomial to get a good approximation.

**Exercise 10.2.** Suppose we want to study the cosine function on the interval $[0, 2\pi]$ and want errors to be less than $10^{-4}$. Which order Taylor approximation (centered at $x_* = 0$) is sufficient?

**Solution:**

With $f(x) = \cos x$ we know that $|f^{(n+1)}(x)| \leq 1$. Thus by the simple estimate we know that the error is less than

$$\frac{2\pi^{n+1}}{(n+1)!}.$$

We want this to be less than $10^{-4}$. A little playing around with numbers leads one to conclude that any $n \geq 23$ will work. So we choose $n = 24$.