TOPIC 8

Convergence tests for not-necessarily positive series

PRINCIPLE (Alternating series).

The series $\sum_{k=*}^{\infty}(-1)^k a_k$ *converges if*

- $a_k \geq 0$, *and*
- $a_k \to 0$ *as* $k \to \infty$.

EXAMPLE. *Consider the series* $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

DEFINITION (Absolute & conditional convergence).

- *If series* $\sum_{k=*}^{\infty} a_k$ *converges, but* $\sum_{k=*}^{\infty} |a_k|$ *does not converge, then we say that* $\sum_{k=*}^{\infty} a_k$ *converges* *__conditionally__*.
- *If series* $\sum_{k=*}^{\infty} a_k$ *converges, but* $\sum_{k=*}^{\infty} |a_k|$ *also converges, then we say that* $\sum_{k=*}^{\infty} a_k$ *converges* *__absolutely__*.

EXAMPLE.

- *The series* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ *converges conditionally.*
- *The series* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ *converges absolutely.*

Principle (Ratio tests for convergence). *We study the series*

$$(\bigstar) \qquad \sum_{k=*}^{\infty} a_k.$$

(1) *If* $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = L < 1$ *for some constant* L*, then* $(\bigstar)$ *converges absolutely.*
(2) *If* $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = L > 1$ *for some constant* L*, then* $(\bigstar)$ *does not converge absolutely.*
(3) *If* $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 1$*, then further investigation is necessary.*

Exercise 8.1. For each of the following series, please do the following:

- First, determine whether the series converges. You must provide reasoning.
- Second, if the series does converge, determine whether it converges absolutely or conditionally. Again, provide rationale.

(1) $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}$

(2) $\sum_{k=1}^{\infty} \frac{k\,2^{k-1}}{k!}$

(3) $\sum_{k=1}^{\infty} \frac{3^k}{k}$

(4) $\sum_{k=1}^{\infty} \frac{(-3)^k}{k}$

(5) $\sum_{k=0}^{\infty} \frac{(-7)^k}{(2k+1)!}$

(6) $\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{2}\right)^k$

(7) $\sum_{k=1}^{\infty} \frac{(-2)^k}{1+3^k}$

(8) $\sum_{k=1}^{\infty} \frac{2k}{5^k}$

(9) $\sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!}$

Solution:

(1) Converges absolutely by ratio test.
(2) Converges absolutely by ratio test.
(3) Diverges as summands do not get small as k grows large.
(4) Diverges as summands do not get small as k grows large.
(5) Converges absolutely by ratio test.
(6) Converges absolutely by comparison to geometric series.
(7) Converges absolutely by comparison to geometric series.
(8) Converges absolutely by ratio test.
(9) Converges absolutely by ratio test.

Exercise 8.2. In this exercise, we examine the convergence of the following series:

$$(\spadesuit) \qquad \sum_{k=1}^{\infty} \left(\frac{k}{4k+1} \right)^k .$$

(1) Show that the partial sums satisfy

$$\begin{aligned} S_n &= \left(\frac{1}{4(1)+1}\right)^1 + \left(\frac{2}{4(2)+1}\right)^2 + \left(\frac{3}{4(3)+1}\right)^3 + \cdots + \left(\frac{n}{4n+1}\right)^n \\ &\leq \left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots + \left(\frac{1}{4}\right)^n \end{aligned}$$

(2) Use comparison to the geometric series to show that the series ($\spadesuit$) converges.

Solution:

(1) The inequality follows from inspecting each term.

(2) The right side of the inequality is the partial sums of a geometric series with ratio $1/4$. Since the geometric series converges, so does ($\spadesuit$).

There are more problems on next page.

Exercise 8.3. (Optional) The purpose of this exercise is to "prove" the following principle.

PRINCIPLE (Root test). *Suppose*

$$\lim_{k\to\infty} |a_k|^{1/k} = L < 1.$$

Then the series

$$\sum_{k=*}^{\infty} a_k$$

converges absolutely.

Construct an argument for the Root Test Principle using the following steps:

(1) Pick some number r between L and 1, meaning a number such that $L < r < 1$. Argue that eventually, the numbers $|a_k|^{1/k}$ are less than r, meaning that there is some point K such that

$$|a_k|^{1/k} < r \qquad \text{whenever } k > K.$$

Re-write this inequality as

$$|a_k| < r^k.$$

(2) Show that the partial sum S_n for the series $\sum_{k=*}^{\infty} |a_k|$ can be written

$$S_n = \sum_{k=*}^{K} |a_k| + |a_{K+1}| + |a_{K+2}| + \cdots + |a_n|.$$

(3) Use the previous expression, together with part (1), to conclude that

$$S_n \leq \sum_{k=*}^{K} |a_k| + r^K \left(r + r^2 + \cdots + r^n\right)$$

(4) Use comparison to geometric series to obtain the desired result.

Exercise 8.4. Use the Root Test Principle to conclude the following series converge absolutely:

$$\sum_{k=1}^{\infty} \left(\frac{2k}{3k+5}\right)^k \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-9)^k}{k^k}.$$

Solution:

For the first series we look at

$$\left[\left(\frac{2k}{3k+5}\right)^k\right]^{1/k} = \left(\frac{2k}{3k+5}\right) \to \frac{2}{5}.$$

Since the limit is less than one we have absolute convergence by the root test.

For the second series we look at

$$\left|\frac{(-9)^k}{k^k}\right|^{1/k} = \frac{9}{k} \to 0.$$

Since the limit is less than one we have absolute convergence by the root test.