TOPIC 8

# Convergence tests for not-necessarily positive series

PRINCIPLE (Alternating series).

*The series* $\sum_{k=*}^{\infty}(-1)^k a_k$ *converges if*

- $a_k \geq 0$, *and*
- $a_k \to 0$ *as* $k \to \infty$.

EXAMPLE. *Consider the series* $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

DEFINITION (Absolute & conditional convergence).

- *If series* $\sum_{k=*}^{\infty} a_k$ *converges, but* $\sum_{k=*}^{\infty} |a_k|$ *does not converge, then we say that* $\sum_{k=*}^{\infty} a_k$ *converges* *<u>conditionally</u>*.
- *If series* $\sum_{k=*}^{\infty} a_k$ *converges, but* $\sum_{k=*}^{\infty} |a_k|$ *also converges, then we say that* $\sum_{k=*}^{\infty} a_k$ *converges* *<u>absolutely</u>*.

EXAMPLE.

- *The series* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ *converges conditionally.*
- *The series* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ *converges absolutely.*

Principle (Ratio tests for convergence). *We study the series*

$$(\bigstar) \qquad \sum_{k=*}^{\infty} a_k.$$

(1) *If* $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = L < 1$ *for some constant* $L$*, then* $(\bigstar)$ *converges absolutely.*

(2) *If* $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = L > 1$ *for some constant* $L$*, then* $(\bigstar)$ *does not converge absolutely.*

(3) *If* $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 1$*, then further investigation is necessary.*

**Exercise 8.1.** For each of the following series, please do the following:

- First, determine whether the series converges. You must provide reasoning.
- Second, if the series does converge, determine whether it converges absolutely or conditionally. Again, provide rationale.

(1) $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}$

(2) $\sum_{k=1}^{\infty} \frac{k\,2^{k-1}}{k!}$

(3) $\sum_{k=1}^{\infty} \frac{3^k}{k}$

(4) $\sum_{k=1}^{\infty} \frac{(-3)^k}{k}$

(5) $\sum_{k=0}^{\infty} \frac{(-7)^k}{(2k+1)!}$

(6) $\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{2}\right)^k$

(7) $\sum_{k=1}^{\infty} \frac{(-2)^k}{1+3^k}$

(8) $\sum_{k=1}^{\infty} \frac{2k}{5^k}$

(9) $\sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!}$

**Exercise 8.2.** In this exercise, we examine the convergence of the following series:

$$(\spadesuit) \qquad \sum_{k=1}^{\infty} \left(\frac{k}{4k+1}\right)^k.$$

(1) Show that the partial sums satisfy

$$\begin{aligned} S_n &= \left(\frac{1}{4(1)+1}\right)^1 + \left(\frac{2}{4(2)+1}\right)^2 + \left(\frac{3}{4(3)+1}\right)^3 + \cdots + \left(\frac{n}{4n+1}\right)^n \\ &\le \left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots + \left(\frac{1}{4}\right)^n \end{aligned}$$

(2) Use comparison to the geometric series to show that the series $(\spadesuit)$ converges.

*There are more problems on next page.*

**Exercise 8.3.** (Optional) The purpose of this exercise is to "prove" the following principle.

PRINCIPLE (Root test). *Suppose*

$$\lim_{k\to\infty} |a_k|^{1/k} = L < 1.$$

*Then the series*

$$\sum_{k=*}^{\infty} a_k$$

*converges absolutely.*

Construct an argument for the Root Test Principle using the following steps:

(1) Pick some number $r$ between $L$ and 1, meaning a number such that $L < r < 1$. Argue that eventually, the numbers $|a_k|^{1/k}$ are less than $r$, meaning that there is some point $K$ such that

$$|a_k|^{1/k} < r \qquad \text{whenever } k > K.$$

Re-write this inequality as

$$|a_k| < r^k.$$

(2) Show that the partial sum $S_n$ for the series $\sum_{k=*}^{\infty} |a_k|$ can be written

$$S_n = \sum_{k=*}^{K} |a_k| + |a_{K+1}| + |a_{K+2}| + \cdots + |a_n|.$$

(3) Use the previous expression, together with part (1), to conclude that

$$S_n \leq \sum_{k=*}^{K} |a_k| + r^K \left(r + r^2 + \cdots + r^n\right)$$

(4) Use comparison to geometric series to obtain the desired result.

**Exercise 8.4.** Use the Root Test Principle to conclude the following series converge absolutely:

$$\sum_{k=1}^{\infty} \left(\frac{2k}{3k+5}\right)^k \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-9)^k}{k^k}.$$