Exercise 7.1. Following the method of the example above, show that each of the following series converges by comparing to a series for which you know convergence.

(1) $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{5^k+1}$

(2) $\displaystyle\sum_{k=0}^{\infty} \frac{3^k-2}{5^k}$

(3) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+\pi k}$

(4) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2+\pi k}$

Solution:

(1) We have $\frac{3^k}{5^k+1} \leq \left(\frac{3}{5}\right)^k$. Thus

$$S_n = \sum_{k=0}^{n} \frac{3^k}{5^k+1} \leq \sum_{k=0}^{n} \left(\frac{3}{5}\right)^k,$$

which converges by geometric series. We conclude that the series converges and

$$\sum_{k=0}^{\infty} \frac{3^k}{5^k+1} \leq \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k = \frac{5}{2}.$$

(2) We have $\frac{3^k-2}{5^k} \leq \frac{3^k}{5^k}$. Thus

$$S_n = \sum_{k=0}^{n} \frac{3^k-2}{5^k} \leq \sum_{k=0}^{n} \left(\frac{3}{5}\right)^k,$$

which converges by geometric series. Thus the series converges and

$$\sum_{k=0}^{\infty} \frac{3^k-2}{5^k} \leq \frac{5}{2}.$$

(3) Since $\pi > 1$ we have $\frac{1}{k^2+\pi k} \leq \frac{1}{k^2+k} = \frac{1}{k} - \frac{1}{k+1}$. Thus we can use telescoping series to conclude that

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2+\pi k} \leq \sum_{k=1}^{n} \left[\frac{1}{k} - \frac{1}{k+1}\right] = 1 - \frac{1}{n+1},$$

which converges as $n \to \infty$. Thus the series converges and

$$\sum_{k=1}^{\infty} \frac{1}{k^2+\pi k} \leq 1.$$

(4) The key is noticing that $\sin^2 k \leq 1$. Then the reasoning from the previous problem shows that the series converges and

$$\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2+\pi k} \leq 1$$

Exercise 7.2. Use comparison to show that each of these series diverges:

$$\sum_{k=1}^{\infty} \frac{2k-1}{k}, \qquad \sum_{k=1}^{\infty} \frac{5 \cdot 3^k}{2^{k-2}}, \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{5+k}{k\sqrt{k}}.$$

Solution:

(1) We expect that the terms $\frac{2k-1}{k}$ is similar to $\frac{2k}{k} = 2$ as k becomes large. Thus we decide to find out where the fraction is greater than half this. Algebraically re-arranging, we have

$$\frac{2k-1}{k} \geq 1 \quad \longleftrightarrow \quad k \geq 2.$$

Thus we have

$$S_n = \sum_{k=1}^{n} \frac{2k-1}{k} = \frac{2(1)-1}{1} + \sum_{k=2}^{n} \frac{2k-1}{k} \geq 1 + \sum_{k=2}^{n} 1 = 2 + (n-1).$$

Therefore the partial sum S_n grows arbitrarily large as n grows and the series diverges.

(2) We re-arrange algebraically to see that

$$\frac{5 \cdot 3^k}{2^{k-2}} = 20 \left(\frac{3}{2}\right)^k$$

Therefore

$$S_n = \sum_{k=1}^{n} \frac{5 \cdot 3^k}{2^{k-2}} = 20 \sum_{k=1}^{n} \left(\frac{3}{2}\right)^k$$

But we know that the geometric series $\sum_{k=*}^{\infty} x^k$ diverges if $x \geq 1$. Thus S_n will grow as $n \to \infty$ and the series diverges.

(3) We expect that for large k we will have $\frac{5+k}{k\sqrt{k}}$ be approximately $\frac{k}{k\sqrt{k}} = \frac{1}{\sqrt{k}}$. Thus we try to find out for which k we have things half this big. Using algebra, we have

$$\frac{5+k}{k\sqrt{k}} \geq \frac{1}{2}\frac{1}{\sqrt{k}} \quad \leftrightarrow \quad k \geq -10.$$

Clearly k is always greater than -10 in our sum, thus

$$S_n = \sum_{k=1}^{n} \frac{5+k}{k\sqrt{k}} \geq \frac{1}{2} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.$$

But the series $\sum_{k=*}^{\infty} \frac{1}{\sqrt{k}}$ diverges (see the p-series below, for example, or compare to an integral). Thus $S_n \to \infty$ as n gets large and the series diverges.

Exercise 7.3. In this exercise you study the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{x}}$.

- Find an integral I_n which bounds from below the partial sum $S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{x}}$.
  That is, find I_n such that
  $$I_n \leq S_n.$$
- Show that $I_n \to \infty$ as $n \to \infty$.
- Conclude that the series diverges.

Solution:

The partial sum is bounded from below as follows

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq I_n = \int_1^{n+1} \frac{1}{\sqrt{x}}\, dx.$$

We now compute the integral

$$I_n = \int_1^{n+1} \frac{1}{\sqrt{x}}\, dx = \left[2\sqrt{x}\right]_1^{n+1} = 2\sqrt{n+1} - 2.$$

Clearly $I_n \to \infty$ as $n \to \infty$ and thus I_n diverges as n grows large. This implies that S_n diverges as n grows large, and thus the series diverges.

Exercise 7.4. Consider now the series $\sum_{k=1}^{\infty} \frac{1}{x^{\pi}}$ using the following steps:

- Find an integral I_n which bounds from above the partial sum $S_n = \sum_{k=1}^{n} \frac{1}{x^{\pi}}$.
  That is, find I_n such that
  $$S_n \leq I_n.$$
- Show that I_n converges to some finite number as $n \to \infty$.
- Conclude that the series ~~diverges~~ converges.

Solution:

We can bound the partial sums as follows

$$S_n = \sum_{k=1}^{n} \frac{1}{x^{\pi}} \leq I_n = 1 + \int_1^{n} \frac{1}{x^{\pi}}\, dx.$$

Computing the integral, we find

$$I_n = 1 + \left[\frac{1}{1-\pi} x^{1-\pi}\right]_1^n = 1 + \frac{n^{1-\pi}}{1-\pi} - \frac{1}{1-\pi}.$$

We have $I_n \to 1 - \frac{1}{1-\pi} = \frac{\pi}{\pi-1}$ as $n \to \infty$. Thus I_n converges, S_n is bounded, and the series converges.

Exercise 7.5. Use integral comparison to determine the convergence of $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$.

Solution:

We can bound the partial sum S_n of series above by the integral

$$I_n = \int_0^n \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^n = \tan^{-1} n.$$

Since $I_n \to \frac{\pi}{2}$ as $n \to \infty$, we see that S_n is bounded and the series converges.

Exercise 7.6. Use integral comparison to determine the convergence of $\sum_{k=1}^{\infty} \frac{1}{e^k}$.

Solution:

We can bound the partial sums as follows

$$S_n = \sum_{k=1}^{n} \frac{1}{e^k} \le I_n = \int_0^n \frac{1}{e^x}\,dx.$$

Computing the integral, we find

$$I_n = \left[-e^{-x}\right]_0^n = 1 - e^{-n}$$

We have $I_n \to 1$ as $n \to \infty$. Thus I_n converges, S_n is bounded, and the series converges.

Exercise 7.7. In this exercise, you examine the so-called "p-series." Here $p > 0$ is some fixed positive number.

(1) Consider the sum $S_n = \sum_{k=1}^{n} \frac{1}{k^p}$. Find integrals $I_n^{\max}$ and $I_n^{\min}$ such that

$$I_n^{\min} \le S_n \le I_n^{\max}.$$

(2) Determine for which values of p the integrals $I_n^{\min}$ and $I_n^{\max}$ converge, and for which values they diverge, as $n \to \infty$.

(3) As a consequence, determine for which values of p the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges, and for which values of p the series diverges.

(4) Restrict attention to the case $p = \frac{3}{2}$.
- Return now to your integrals $I_n^{\min}$ and $I_n^{\max}$.
- Let $I^{\min}$ and $I^{\max}$ be the limits of these sequences (i.e. $I_n^{\min} \to I^{\min}$, etc.).
- Find numerical values for $I^{\min}$ and $I^{\max}$.
- We know (why?) that $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges to some limit L.
- What is the relationship between $I^{\min}$, $I^{\max}$, and L?

(5) Now you are to repeat the previous part for generic p.
- Restrict attention to p such that the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges.
- Find, using integral comparison, numbers $I^{\min}$ and $I^{\max}$ as before.

- What is the relationship between $I^{\min}$, $I^{\max}$, and L?

Solution:

We studies the case $p = 1$ in class and concluded that that series diverges; thus we only consider other value of p in the work below. Notice that many of the formulas below don't work for $p = 1$, making this a special case which needs to be treated separately.

(1) With $S_n = \sum_{k=1}^{n} \frac{1}{k^p}$ we have

$$\underbrace{\int_1^{n+1} \frac{1}{x^p}\,dx}_{I_n^{\min}} \le S_n \le \underbrace{1 + \int_1^{n} \frac{1}{x^p}\,dx}_{I_n^{\max}}$$

(2) We compute

$$I_n^{\min} = \left[\frac{1}{1-p}x^{1-p}\right]_1^{n+1} = \frac{1}{p-1} - \frac{(n+1)^{1-p}}{p-1}$$

$$I_n^{\max} = 1 + \left[\frac{1}{1-p}x^{1-p}\right]_1^{n} = 1 + \frac{1}{p-1} - \frac{(n)^{1-p}}{p-1}$$

Thus if $p < 1$ we see that the integrals diverge, while if $p > 1$ the integrals converge.

(3) Therefore we conclude from integral comparison that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \le 1$.

(4) With $p = \frac{3}{2}$ we have

$$I_n^{\min} \to 2 = I^{\min} \quad \text{and} \quad I_n^{\max} \to 3 = I^{\max}.$$

We know that the series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges to some limit L by our previous work. We now know that

$$\underbrace{2}_{I^{\min}} \le L \le \underbrace{3}_{I^{\max}}.$$

(5) In the generic case we have

$$I_n^{\min} \to \frac{1}{p-1} = I^{\min} \quad \text{and} \quad I_n^{\max} \to \frac{p}{p-1} = I^{\max}.$$

This leads to

$$\underbrace{\frac{1}{p-1}}_{I^{\min}} \le L \le \underbrace{\frac{p}{p-1}}_{I^{\max}}.$$