TOPIC 7

Comparison for positive sequences

- Direct comparison
- Limit comparison
- Sequences of integrals $\rightsquigarrow$ improper integrals
- Integral comparison

PRINCIPLE (Bounded, monotone).

If a sequence S_n is

- *bounded above, meaning $S_n \leq M$ for some number M, and*
- *monotone increasing, meaning that $S_{n+1} \geq S_n$ for each n*

then we conclude that S_n converges to some limit L as $n \to \infty$.

Application to series:

- *If the sequence of summands a_k are non-negative, meaning that $a_k \geq 0$ for each k, then we know that the partial sums $S_n = \sum_{k=*}^{n} a_k$ will be monotone increasing. Thus once we know S_n is bounded, we obtain convergence.*
- *Typically, we show boundedness by comparison.*

PRINCIPLE (Comparison).

Suppose we have two sequences of summands a_k and b_k which satisfy

$$0 \leq a_k \leq b_k \qquad \textit{for each } k.$$

Then

(1) *If $\displaystyle\sum_{k=*}^{\infty} b_k$ converges, then $\displaystyle\sum_{k=*}^{\infty} a_k$ converges.*

(2) *If $\displaystyle\sum_{k=*}^{\infty} a_k$ diverges, then $\displaystyle\sum_{k=*}^{\infty} b_k$ diverges.*

PRINCIPLE (Summand limit).

Suppose that $a_k \to L > 0$ *as* $k \to \infty$. *Then the series* $\sum_{k=*}^{\infty} a_k$ *diverges.*

PRINCIPLE (Integral comparison).

Suppose we have a positive function $f(x)$ *such that* $f(k) = a_k$. *Then*

$$\int_K^{N+1} f(x)\,dx \le \sum_{k=K}^{N} a_k \le \int_{K-1}^{N} f(x)\,dx.$$

In particular,

- *if* $I_N = \int_*^N f(x)\,dx$ *diverges, so does* $\sum_{k=*}^{\infty} a_k$,
- *if* $I_N = \int_*^N f(x)\,dx$ *converges, so does* $\sum_{k=*}^{\infty} a_k$, *and*
- *if* $\sum_{k=*}^{\infty} a_k$ *converges, we can estimate the remainder ...*

EXAMPLE. *Show that the series* $\sum_{k=2}^{\infty} \frac{2^k - 1}{5^k}$ *converges.*

We first compare, noting that

$$\frac{2^k - 1}{5^k} \le \frac{2^k}{5^k} = \left(\frac{2}{5}\right)^k.$$

Since $\sum_{k=*}^{\infty} \left(\frac{2}{5}\right)^k$ *converges, we know from the comparison principle that* $\sum_{k=2}^{\infty} \frac{2^k - 1}{5^k}$ *converges as well.*

Exercise 7.1. Following the method of the example above, show that each of the following series converges by comparing to a series for which you know convergence.

(1) $\sum_{k=0}^{\infty} \frac{3^k}{5^k + 1}$

(2) $\sum_{k=0}^{\infty} \frac{3^k - 2}{5^k}$

(3) $\sum_{k=1}^{\infty} \frac{1}{k^2 + \pi k}$

(4) $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2 + \pi k}$

Exercise 7.2. Use comparison to show that each of these series diverges:

$$\sum_{k=1}^{\infty} \frac{2k - 1}{k}, \qquad \sum_{k=1}^{\infty} \frac{5 \cdot 3^k}{2^{k-2}}, \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{5 + k}{k\sqrt{k}}.$$

Exercise 7.3. In this exercise you study the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{x}}$.

- Find an integral I_n which bounds from below the partial sum $S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{x}}$. That is, find I_n such that
$$I_n \leq S_n.$$
- Show that $I_n \to \infty$ as $n \to \infty$.
- Conclude that the series diverges.

Exercise 7.4. Consider now the series $\sum_{k=1}^{\infty} \frac{1}{x^{\pi}}$ using the following steps:

- Find an integral I_n which bounds from above the partial sum $S_n = \sum_{k=1}^{n} \frac{1}{x^{\pi}}$. That is, find I_n such that
$$S_n \leq I_n.$$
- Show that I_n converges to some finite number as $n \to \infty$.
- Conclude that the series diverges.

Exercise 7.5. Use integral comparison to determine the convergence of $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$.

Exercise 7.6. Use integral comparison to determine the convergence of $\sum_{k=1}^{\infty} \frac{1}{e^k}$.

Exercise 7.7. In this exercise, you examine the so-called "p-series." Here $p > 0$ is some fixed positive number.

(1) Consider the sum $S_n = \sum_{k=1}^{n} \frac{1}{k^p}$. Find integrals $I_n^{\max}$ and $I_n^{\min}$ such that
$$I_n^{\min} \leq S_n \leq I_n^{\max}.$$
(2) Determine for which values of p the integrals $I_n^{\min}$ and $I_n^{\max}$ converge, and for which values they diverge, as $n \to \infty$.
(3) As a consequence, determine for which values of p the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges, and for which values of p the series diverges.
(4) Restrict attention to the case $p = \frac{3}{2}$.
  - Return now to your integrals $I_n^{\min}$ and $I_n^{\max}$.
  - Let $I^{\min}$ and $I^{\max}$ be the limits of these sequences (i.e. $I_n^{\min} \to I^{\min}$, etc.).
  - Find numerical values for $I^{\min}$ and $I^{\max}$.
  - We know (why?) that $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges to some limit L.
  - What is the relationship between $I^{\min}$, $I^{\max}$, and L?

(5) Now you are to repeat the previous part for generic p.

- Restrict attention to p such that the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges.
- Find, using integral comparison, numbers $I^{\min}$ and $I^{\max}$ as before.
- What is the relationship between $I^{\min}$, $I^{\max}$, and L?

Exercise 7.8. In this problem we increase our knowledge of the p-series, focusing on the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}. \tag{$\bigstar$}$$

(1) Explain why we know the series ($\bigstar$) converges.

(2) We write the series as

$$\sum_{k=1}^{\infty} \frac{1}{k^{5/2}} = \underbrace{\sum_{k=1}^{n} \frac{1}{k^{5/2}}}_{S_n} + \underbrace{\sum_{k=n+1}^{\infty} \frac{1}{k^{5/2}}}_{R_n}$$

and consider the partial sums for R_n, which we denote as R_n^N. Specifically, we have

$$R_n^N = \sum_{k=n+1}^{N} \frac{1}{k^{5/2}}.$$

Find integrals $I_n^{N,\max}$ and $I_n^{N,\min}$ such that

$$I_n^{N,\min} \leq R_n^N \leq I_n^{N,\max}$$

(3) Show that there exists numbers (depending on n of course) $I_n^{\max}$ and $I_n^{\min}$ such that

$$I_n^{N,\min} \to I_n^{\min} \quad \text{and} \quad I_n^{N,\max} \to I_n^{\max} \qquad \text{as} \quad N \to \infty.$$

(4) Assemble the knowledge you have gained in to an expression of the form

$$S_n + \boxed{?} \leq \sum_{k=1}^{\infty} \frac{1}{k^{5/2}} \leq S_n + \boxed{??}.$$

Conclude that S_n approximates the series ($\bigstar$) with error no larger than $\boxed{??} - \boxed{?}$.

(5) Suppose I approximate the series ($\bigstar$) by S_5. How good is the approximation? (Meaning, how large might the error be?)

(6) Suppose we want to approximate the series ($\bigstar$) to within an accuracy of 0.001. What value of n assures us that S_n is a sufficient approximate?

Exercise 7.9.

(1) Find the partial sum S_5 for the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$.

(2) How close is S_5 to the limit of the series?

(3) What must n be in order for S_n to be within 0.0001 of the true limit of the series?

Exercise 7.10. (Optional, fun) Use technology to investigate the limit of the sequence

$$D_n = \left(\sum_{k=1}^{n} \frac{1}{k} \right) - \ln(n).$$

Then read more about this sequence at

http://mathworld.wolfram.com/Euler-MascheroniConstant.html

Report on what you learned.

Exercise 7.11. (Optional; challenging)

(1) Suppose $x > 1$ is a fixed number. Then we can consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^x},$$

which converges to some number. We can view this process as a function of x – this function is called the *Riemann zeta function* and denoted

$$\zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}.$$

(Here ζ is the Greek letter 'zeta'.) In class we extensively analyzed $\zeta(2)$, in Exercise 7.8 you studied $\zeta(5/2)$, and in Exercise 7.9 you studied $\zeta(3)$.
Argue, based on your knowledge of convergence, that $\lim_{x\to 1^+} \zeta(x) = +\infty$. Use this to draw a graph of $\zeta(x)$ on the domain $x > 1$.

(2) It is also possible to put complex numbers of the form $x = a + bi$ in to the zeta function. Here i is the number such that $i^2 = -1$. (In order to learn more about how to do calculus using complex numbers take our course "Complex Variables" next year.) With complex inputs, it is possible that $\zeta(a + bi) = 0$. Make a list of all such complex numbers $a + bi$.[1] For a hint see this webpage:

http://www.claymath.org/millenium-problems/riemann-hypothesis

[1] If you make progress, please kindly remember your calculus 2 teacher. . .