TOPIC 6

Infinite series 1: Geometric and telescoping series

Main ideas.

- Convergence and divergence: general definitions and intuitions
- Geometric series: $\sum_{k=0}^{\infty} r^k$
- Telescoping series $\sum_{k=*}^{\infty} \frac{1}{\text{quadratic}}$

Exercises.

Exercise 6.1. For each of the series below, please

- Write out the first few partial sums S_1, S_2, S_3
- Write out a general formula for S_n
- Determine if the series converges. If the series is convergent, to what does it converge? In either case, explain your reasoning.

(1) $$\sum_{k=0}^{\infty} (-1)^k \frac{2^k}{3^k}$$

(2) $$\sum_{k=0}^{\infty} \frac{5^k}{4^k}$$

(3) $$\sum_{k=0}^{\infty} \frac{2}{4^k}$$

(4) $$\sum_{k=0}^{\infty} \left(\frac{1}{3^k} + \frac{5}{6^k} \right)$$

(5) $$\sum_{k=1}^{\infty} \frac{1}{k^2 + 2k}$$

(6) $$\sum_{k=3}^{\infty} \frac{1}{k^2 - 4}$$

(7) $$\sum_{k=1}^{\infty} \frac{k}{k^2 + k}$$

Note: There are more problems on the next page!

Exercise 6.2. Write each of the following series in terms "standard" geometric series. Then determine whether it converges or not.

Example: Suppose we are given the series $\sum_{k=2}^{\infty} \frac{2^{2k+1}}{3^k}$. First we re-write as

$$\sum_{k=2}^{\infty} \frac{2^{2k+1}}{3^k} = \sum_{k=2}^{\infty} \frac{2 \cdot 4^k}{3^k} = 2\sum_{k=2}^{\infty} \left(\frac{4}{3}\right)^k = 2\sum_{k=2}^{\infty} \left(\frac{4}{3}\right)^{k-2} \left(\frac{4}{3}\right)^2 = \frac{32}{9}\sum_{k=2}^{\infty} \left(\frac{4}{3}\right)^{k-2}.$$

Next we define a new counter $l = k - 2$. Notice that when $k = 2$ our new counter has $l = 0$. Thus we have

$$\sum_{k=2}^{\infty} \frac{2^{2k+1}}{3^k} = \frac{32}{9} \sum_{l=0}^{\infty} \left(\frac{4}{3}\right)^l.$$

The series does not converge because the ratio $\frac{4}{3}$ is greater than 1.

(1) $\sum_{n=2}^{\infty} \frac{3^n}{4^n}$

(2) $\sum_{n=3}^{\infty} (-1)^n \frac{2}{3^n}$

(3) $\sum_{n=1}^{\infty} \frac{2}{3^{n+2}}$

(4) $\sum_{n=3}^{\infty} (-1)^n \frac{3^n}{2 \cdot 4^n}$

(5) $\sum_{n=0}^{\infty} \left(\frac{1}{3^{n-1}} - \frac{2}{9^n}\right)$

(6) $\sum_{n=-1}^{\infty} \frac{5^{n-3}}{6^{2n-1}}$

Exercise 6.3. (Challenge) Recall that the geometric sum formula gives us a nice expression for the quantity

$$1 + x + x^2 + \cdots + x^n.$$

Use this to find a nice formula for the quantity

$$1 + 2x + 3x^n + \cdots + nx^{n-1}.$$

Use your formula to analyze the convergence of the series

$$\sum_{k=1}^{\infty} k \left(\frac{1}{3}\right)^{n-1}.$$