TOPIC 4

Sequences

Main ideas.

- Functions & sequences
- Features of functions and sequences
- Limits & convergence for sequences

Exercises.

Exercise 4.1. The point of this problem is for you to become familiar with some vocabulary surrounding calculus.

(1) Describe the technical meaning of the words *function*, *domain*, *graph* as they appear in the context of this course.
(2) What is the difference between the concepts *function* and *equation*? Give an illustrative example.
(3) What is a *sequence*? How does this concept relate to that of a function?
(4) Compare and contrast the concepts *discrete* and *continuous*.
(5) Give examples of both continuous functions and (discrete) sequences which are. . .
   . . . increasing.
   . . . decreasing.
   . . . bounded from above.
   . . . bounded from below.
   . . . unbounded.
(6) Give both a "careful" definition of the limit of a sequence, and also an "intuitive" description of the limit. Illustrate with an example.
(7) Give three examples of convergent sequences and three examples of divergent sequences.

Exercise 4.2. For each sequence below, do the following:

- Write out the first few terms of the sequence.
- Plot the first few terms of the sequence.
- Decide if the sequences is increasing/decreasing, bounded/unbounded, etc. Give rationale when possible.
- Determine if the sequence is convergent or divergent. Provide the best rationale you can.

(1) $a_n = \frac{(-1)^n}{n}$
(2) $a_n = \cos(n\pi)$
(3) $a_n = \left(1 + \frac{1}{n}\right)^n$
(4) $a_n = \ln(2n+1) - \ln(n)$
(5) $a_0 := 1, \quad a_{n+1} := \frac{1}{2}\left(a_n + \frac{4}{a_n}\right)$
(6) $a_0 := 1, \quad a_{n+1} := (n+1) \cdot a_n$
(7) $a_0 := 0, \quad a_{n+1} := a_n + \frac{1}{2^n}$
(8) $a_n = \frac{1}{n!}$
(9) $a_n = \frac{e^n}{n!}$
(10) $a_0 := 0, \quad a_1 := 1, \quad a_{n+1} := 5a_n - 6a_{n-1}$
(11) $a_n := \int_{-\pi}^{\pi} x \sin(nx)\, dx$
(12) a_n is the value of the Riemann sum for the function $f(x) = x^2$ over the interval $[0,1]$ obtained by using n subdivisions and right endpoints as sample points.

Exercise 4.3.
(1) Suppose we define the sequence S_n by $S_n = 1 + 2 + 3 + \cdots + n$. Do you think S_n converges as $n \to \infty$? Explain.
(2) Suppose now that we define a new sequence $T_n = \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}$. Do you think T_n converges as $n \to \infty$? Explain.
(3) Next, define R_n by $R_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$. Do you think R_n converges as $n \to \infty$? Explain.
(4) Show (using Gauß' trick) that $S_n = \frac{n(n+1)}{2}$. Use this to "check" your previous responses.

Exercise 4.4. Define the sequence S_n by $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$.
(1) Find a formula for S_n. [Hint, it is always the case that $S_n < 1$.]
(2) Use your formula to conclude that S_n converges. What does it converge to?

Exercise 4.5. Suppose we have a sequence a_n such that $a_n \to 0$ as $n \to \infty$.
(1) Describe the behavior of the sequence $b_n = \dfrac{1}{a_n}$ as $n \to \infty$? Explain your reasoning.
(2) Describe the behavior of the sequence $b_n = \sin(a_n)$ as $n \to \infty$? Explain your reasoning.
(3) Describe the behavior of the sequence $b_n = e^{a_n}$ as $n \to \infty$? Explain your reasoning.
(4) Describe the behavior of the sequence $b_n = \dfrac{a_n}{3 + a_n}$ as $n \to \infty$? Explain your reasoning.

Exercise 4.6. Suppose we have a sequence a_n such that $a_n \to +\infty$ as $n \to \infty$.
(1) Describe the behavior of the sequence $b_n = \dfrac{1}{a_n}$ as $n \to \infty$? Explain your reasoning.
(2) Describe the behavior of the sequence $b_n = \dfrac{2a_n + 3}{a_n}$ as $n \to \infty$? Explain your reasoning.
(3) Describe the behavior of the sequence $b_n = \dfrac{2a_n}{3 + a_n}$ as $n \to \infty$? Explain your reasoning.