

TOPIC 3

Taylor polynomials

Main ideas.

- Linear approximating functions: Review
- Approximating polynomials
- Key formulas:

$$P_n(x) = a_0 + a_1(x - x_*) + \cdots + a_n(x - x_*)^n \quad \text{where} \quad a_k = \frac{f^{(k)}(x_*)}{k!}$$
$$P_n(x_* + \Delta x) = a_0 + a_1(\Delta x) + \cdots + a_n(\Delta x)^n$$

Mathematica code. Here is some basic mathematica code for plotting functions.

- (1) Plot the function $f(x) = \sin(3x)$.

```
Plot[Sin[3 x], {x, -Pi, Pi}]
```

- (2) Plot the function $f(x) = \sin(3x)$ in red, and the line $y = 3x$ in blue.

```
Plot[{Sin[3 x], 3 x}, {x, -Pi, Pi}, PlotRange -> {-3, 3},  
PlotStyle -> {Red, Blue}]
```

Here is some more sophisticated Mathematica code for exploring Taylor series.

- (1) The exponential function $f(x) = e^{2x}$

```
Manipulate[  
  Plot[{Exp[2 x], Sum[((2 x)^k)/(k!), {k, 0, n}]},  
    {x, -1, 1}, PlotRange -> {0, 4},  
    PlotStyle -> {{Thick, Red}, {Thick, Blue, Dashed}}],  
  {n, 1, 10, 1}]
```

- (2) The function $f(x) = \cos x$. (Note that in this case n does not correspond to the degree of the polynomial!)

```
Manipulate[  
  Plot[{Cos[x], Sum[((-1)^k (x)^(2 k))/((2k)!), {k, 0,n}]},  
    {x, -3 Pi, 3 Pi}, PlotRange -> {-2, 2},  
    PlotStyle -> {{Thick, Red}, {Thick, Blue, Dashed}}],  
  {n, 1, 10, 1}]
```

Exercises.

Exercise 3.1. Write down the fourth-order Taylor polynomials, centered at $x_* = 0$, for the following functions:

- | | |
|----------------|---------------------|
| (1) e^x | (3) $\sqrt{1+x}$ |
| (2) $\ln(1+x)$ | (4) $\frac{1}{1+x}$ |

Can you find expressions for the “general” polynomials $P_n(x)$?

Exercise 3.2.

- (1) Consider $f(x) = \sin x$. Show that $f^{(k)}(0) = 0$ when k is even, and that when $k = 2l + 1$ is odd we have $f^{(2l+1)}(0) = (-1)^l$. Conclude that the Taylor polynomial, centered at $x_* = 0$, for $\sin x$

$$x + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^l}{(2l+1)!}x^{2l+1} + \cdots$$

- (2) Show that the Taylor polynomial for the cosine function, centered at $x_* = 0$, is given by

$$1 + \frac{-1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{(-1)^l}{(2l)!}x^{2l} + \cdots$$

- (3) Suppose f is an *even function*, meaning that $f(x) = f(-x)$ for all x . Show that this implies that $f^{(k)}(0) = 0$ whenever k is odd. [Hint: Take k derivatives of the identity defining even-ness.] What can you conclude about the Taylor polynomials centered at $x_* = 0$?

Exercise 3.3. Using the polynomials you found in Exercises 3.1 and 3.2, find the following Taylor polynomials. [Do not do any extra work!]

- (1) Find the fourth-order Taylor polynomial for e^{3x} , centered at $x_* = 0$.
- (2) Find the fourth-order Taylor polynomial for $\ln(x)$, centered at $x_* = 1$.
- (3) Find the fourth-order Taylor polynomial for $\cos(5x)$, centered at $x_* = 0$.
- (4) Find the eighth-order Taylor polynomial for $\sqrt{1+4x^2}$, centered at $x_* = 0$.

Exercise 3.4. In this exercise, we explore the Taylor polynomial approximation for $f(x) = \frac{1}{1-x}$.

- (1) Compute several derivatives of f and conclude that $f^{(k)}(0) = k!$.
- (2) Write down the formula for the Taylor polynomial $P_n(x)$ for f , centered at $x_* = 0$.
- (3) What is $P_n(1)$? (This will depend on n , of course.) What happens to $P_n(1)$ as n gets large?
- (4) What is $f(1)$? In what sense does this “agree” with your response to the previous question?
- (5) Find a formula for $P_n(-1)$. (Again, your formula will depend on n .) What happens as n gets large?

- (6) What is $f(-1)$? In what sense does this “agree” with your response to the previous question?
- (7) Compare $P_n(2)$ with $f(2)$ as n gets large. Are the values close to one another? Explain.
- (8) Use the following code to compare the Taylor polynomials to f .

```
Manipulate[
  Plot[{1/(1 - x), Sum[x^k, {k, 0, n}]},
    {x, -2, 3}, Exclusions -> {1}, PlotRange -> {-5, 5},
    PlotStyle -> {{Thick, Red}, {Thick, Blue, Dashed}},
    {n, 1, 10, 1}]
```

To what extent do the polynomials do a good job approximating the function; to what extent do the polynomials do a poor job?

Exercise 3.5. In the previous problem, we saw that the Taylor polynomials $P_n(x)$ approximated the function $f(x)$ very well for a certain range of x , but did not do a good job approximating the function for other x . One might conjecture that this is because the function f considered has an asymptote. In this exercise we see that the Taylor polynomials might not do a good job approximating a function, *even if that function has no vertical asymptotes!*

- (1) Consider the function $g(x) = \frac{1}{1+x^2}$. Use Exercise 3.1 to show that the Taylor polynomial approximating g near $x_* = 0$ are given by

$$P_n(x) = 1 - x^2 + x^4 - x^6 + \dots$$

- (2) Use the following code to explore how well the Taylor polynomials approximate g . Write a few sentences explaining your findings.

```
Manipulate[
  Plot[{1/(1 + x^2), Sum[(-x^2)^k, {k, 0, n}]},
    {x, -2, 3}, Exclusions -> {1}, PlotRange -> {-1, 2},
    PlotStyle -> {{Thick, Red}, {Thick, Blue, Dashed}},
    {n, 1, 10, 1}]
```