Preparing for the midterm exam

Use the following problems to study for the midterm exam:

(1) What does it mean for a sequence A_k to *converge* to L as $k \to \infty$? Give an example of a sequence which converges, and an example of a sequence which diverges.

(2) Determine if the following sequences converge; if the sequence converges, to what does it converge?

(a) $b_k = \frac{2k}{3k+1}$
(b) $c_k = \frac{2k^2}{3k+1}$
(c) $r_k = \left(\frac{3}{4}\right)^k$
(d) $s_k = (-1)^k$
(e) $g_k = \ln(2k+1)$
(f) $f_k = \frac{1}{k} - \frac{1}{k+1}$
(g) $l_k = \ln(k+1) - \ln(k)$
(h) $p_k = \tan^{-1}(k)$

(3) Find a simple formula/expression for the following sums:

(a) $2 + 4 + 6 + \cdots + 2014$
(b) $3 + 6 + 12 + 24 + \cdots + 768$
(c) $1 + 8 + 27 + \cdots + 100^3$
(d) $\displaystyle\sum_{k=0}^{25} \left(\frac{4}{3}\right)^k$
(e) $\displaystyle\sum_{k=1}^{10} \left(1 + k^2\right)$
(f) $\displaystyle\sum_{k=1}^{12} \frac{2 \cdot 3^k}{5^{2k}}$
(g) $\displaystyle\sum_{k=1}^{42} \frac{1}{k^2 + 4k}$

(4) Answer the following questions about the series $\displaystyle\sum_{k=1}^{\infty} a_k$:
(a) What do we mean by the *partial sum* S_n?
(b) What does it mean for the *series to converge to* L?
(c) What would it mean for the *series to converge absolutely*?

(5) To what value do the following series converge?

(a) $\displaystyle\sum_{k=0}^{\infty} 0.7^k$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2 + k}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{2(-3)^k}{5^{2k}}$

(6) What is a *geometric series*? How do we know when a geometric series converges? Give an example of a convergent geometric series and an example of a divergent geometric series.

(7) What is a *telescoping series*? Give an example.

(8) (a) Explain how to use integral comparison to determine whether a series converges or diverges.

(b) Use integrals to determine whether the following series converge or diverge.

(i) $\sum_{k=0}^{\infty} e^k$ (iii) $\sum_{k=1}^{\infty} \frac{2}{\sqrt{k}}$ (v) $\sum_{k=0}^{\infty} \frac{1}{1+k^2}$

(ii) $\sum_{k=0}^{\infty} e^{-k}$ (iv) $\sum_{k=1}^{\infty} \frac{2}{(\sqrt{k})^3}$ (vi) $\sum_{k=0}^{\infty}(2k+1)$

(9) For which values of p does this series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converge? Explain.

(10) (a) Explain how to use integrals to determine an interval in which the limit of a series lies.

(b) Use integrals to find numbers m and M such that

$$m \leq \sum_{k=1}^{\infty} \frac{1}{k^5} \leq M.$$

(c) Explain how to use integrals to estimate how well S_n approximates the limiting value of a convergent series.

(d) Find a value of n such that the difference between

$$\sum_{k=1}^{n} \frac{1}{k^5} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^5}$$

is less than one-millioneth.

(11) (a) Explain how to use comparison of series to show that a series converges or diverges.

(b) Use comparison with a p-series in order to determine whether the following series converge:

(i) $\sum_{k=1}^{\infty} \frac{2}{3k+1}$ (iii) $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

(ii) $\sum_{k=1}^{\infty} \frac{2}{k^3+1}$ (iv) $\sum_{k=1}^{\infty} \frac{2}{\sqrt{3k+1}}$

(12) (a) Explain what an *alternating series* is. How do we know whether an alternating series converges?

(b) Give an example of a convergent alternating series and an example of a divergent alternating series.

(13) (a) Explaing the difference between *absolute convergence* and *conditional convergence* for a series.

(b) Give an example of a conditionally convergent series.

(c) Give an example of an absolutely convergent series.

(d) Determine whether the following are absolutely convergent, conditionally convergent, or divergent:

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

(iv) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$

(v) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{k^5}$

(vi) $\displaystyle\sum_{k=1}^{\infty} \frac{(-2)^k}{k!}$

(14) (a) What do we mean by the n^{th} order Taylor polynomial for function $f(x)$ near x_*?

(b) Find the 3^{rd} order Taylor polynomial for function $f(x) = e^{4x}$ near $x_* = 0$.

(c) Find the 3^{rd} order Taylor polynomial for function $f(x) = e^{4x}$ near $x_* = 5$.

(d) Find the 5^{th} order Taylor polynomial for function $f(x) = \sin x$ near $x_* = 0$.

(e) Find the 6^{th} order Taylor polynomial for function $f(x) = \cos x$ near $x_* = 0$.

(f) Find the 3^{rd} order Taylor polynomial for function $f(x) = \ln x$ near $x_* = 1$.

(g) Find the 3^{rd} order Taylor polynomial for function $f(x) = \ln(1 + x)$ near $x_* = 0$.

(15) (a) Find the 3^{th} order Taylor polynomial for function $g(x) = xe^x$ near $x_* = 0$. Call this polynomial $p(x)$.

(b) Find the 2^{nd} order Taylor polynomial for $f(x) = e^x$ near $x_* = 0$. Call this polynomial $q(x)$.

(c) Observe that $p(x) = x\,q(x)$. Make a general hypothesis about the relationship between the Taylor polynomials for $f(x)$ and $g(x) = x\,f(x)$...