Instructions: Complete the following problems on the paper I provide. Please write on only one side of each page. You must show legible work in order to receive credit. If you are not confident in some result, you will receive more credit if you make a note of this and comment on where you might be going wrong or on alternate approaches you might try.

Problem 1. (7 minutes)

Compute $\displaystyle\sum_{k=3}^{\infty} \frac{(-3)^k}{5^{2k}}$.

Solution:

Re-index the sum with $l = k - 3$ and use the geometric series formula to obtain

$$\sum_{k=3}^{\infty} \frac{(-3)^k}{5^{2k}} = \sum_{l=0}^{\infty} \frac{(-3)^{l+3}}{25^{l+3}} = \frac{-27}{25^3} \sum_{l=0}^{\infty} \left(\frac{-3}{25}\right)^l = \frac{-27}{25^3} \frac{1}{1 - \left(\frac{-3}{25}\right)} = \boxed{-\frac{27}{25^2 \cdot 28}}$$

Problem 2. (7 minutes)

Does the series $\displaystyle\sum_{k=1}^{\infty} \frac{3}{7k^2 + 2}$ converge? Explain.

Solution:

For the partial sum S_n we have

$$S_n = \sum_{k=1}^{n} \frac{3}{7k^2 + 2} \le \sum_{k=1}^{n} \frac{3}{7k^2} \le \frac{3}{7} \sum_{k=1}^{n} \frac{1}{k^2}.$$

Since $\sum_{k=*}^{\infty} \frac{1}{k^2}$ converges by p-series, the comparison principle tells us that the original series converges.

Problem 3. (12 minutes)

Find an integer n so that the difference between

$$\sum_{k=1}^{n} \frac{1}{k^4} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^4}$$

is less than 10^{-6}.

Solution:

The difference between the two is the remainder

$$R_n = \sum_{k=n+1}^{\infty} \frac{1}{k^4}.$$

We over-estimate the remainder by an integral:

$$R_n \leq \lim_{N\to\infty} \int_n^N \frac{1}{x^4}\,dx = \lim_{N\to\infty} \left[-\frac{1}{3}x^{-3}\right]_n^N = \lim_{N\to\infty} \left[-\frac{1}{3}\frac{1}{N^3} + \frac{1}{3}\frac{1}{n^3}\right] = \frac{1}{3n^3}.$$

Thus it suffices to choose n so that

$$\frac{1}{3n^3} \leq 10^{-6} \quad \leftrightarrow \quad n^3 \geq \frac{1}{3}10^6 \quad \leftrightarrow \quad n \geq \frac{100}{\sqrt[3]{3}}.$$

Choosing $n = 100$, for example, is clearly sufficient.

Problem 4. (24 minutes)

In this problem we study the series $\sum_{k=1}^{\infty} \frac{x^k}{\sqrt{k}}$ for various values of x.

1. Suppose we replace x by -1. Does the series converge?
2. Suppose we replace x by 1. Does the series converge?
3. Suppose x is a number such that $|x| < 1$ (i.e. $-1 < x < 1$). Does the series converge?
4. Suppose x is a number such that $|x| > 1$ (i.e. either $x < -1$ or $1 < x$). Does the series converge?

Solution:

1. If we replace x by -1 we obtain the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$, which converges by the alternating series principle.
2. If we replace x by 1 we obtain the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, which diverges due to the p-series principle.
3. Supposing that $|x| < 1$ we now analyze the series using the Ratio Test Principle:
$$\text{Ratio} = \frac{\left|\frac{x^{k+1}}{\sqrt{k+1}}\right|}{\left|\frac{x^k}{\sqrt{k}}\right|} = \frac{\sqrt{k}}{\sqrt{k+1}}|x| \to |x|.$$
Thus by the ratio test we have that the series converges when $|x| < 1$.
4. When $|x| > 1$ the ratio computation above shows that the series cannot converge absolutely. Furthermore, the terms in the series do not get small as $k \to \infty$. Thus the series does not converge at all.

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Problem 1. (7 minutes)

Compute $\sum_{k=3}^{\infty} \frac{(-2)^k}{3^{2k}}$.

Solution:

Re-index the sum with $l = k-3$ and use the geometric series formula to obtain

$$\sum_{k=3}^{\infty} \frac{(-2)^k}{3^{2k}} = \sum_{l=0}^{\infty} \frac{(-2)^{l+3}}{9^{l+3}} = \frac{-8}{9^3} \sum_{l=0}^{\infty} \left(\frac{-2}{9}\right)^l = \frac{-8}{9^3} \frac{1}{1-\left(\frac{-2}{9}\right)} = \boxed{-\frac{8}{9^2 \cdot 11}}$$

Problem 2. (7 minutes)

Does the series $\sum_{k=1}^{\infty} \frac{5}{7k^2+3}$ converge? Explain.

Solution:

For the partial sum S_n we have

$$S_n = \sum_{k=1}^{n} \frac{5}{7k^2+3} \leq \sum_{k=1}^{n} \frac{5}{7k^2} \leq \frac{5}{7} \sum_{k=1}^{n} \frac{1}{k^2}.$$

Since $\sum_{k=*}^{\infty} \frac{1}{k^2}$ converges by p-series, the comparison principle tells us that the original series converges.

Problem 3. (12 minutes)

Find an integer n so that the difference between

$$\sum_{k=1}^{n} \frac{1}{k^3} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^3}$$

is less than 10^{-4}.

Solution:

The difference between the two is the remainder

$$R_n = \sum_{k=n+1}^{\infty} \frac{1}{k^3}.$$

We over-estimate the remainder by an integral:

$$R_n \leq \lim_{N\to\infty} \int_n^N \frac{1}{x^3}\, dx = \lim_{N\to\infty} \left[-\frac{1}{2} x^{-2} \right]_n^N = \lim_{N\to\infty} \left[-\frac{1}{2}\frac{1}{N^2} + \frac{1}{2}\frac{1}{n^2} \right] = \frac{1}{2n^2}.$$

Thus it suffices to choose n so that

$$\frac{1}{2n^2} \leq 10^{-4} \quad \leftrightarrow \quad n^2 \geq \frac{1}{2} 10^4 \quad \leftrightarrow \quad n \geq \frac{100}{\sqrt{2}}.$$

Choosing $n = 100$, for example, is clearly sufficient.

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