(2)

(a) $b_k = \frac{2k}{3k+1} = \frac{2}{3 + 1/k}$

Since $\frac{1}{k} \to 0$, $b_k \to \frac{2}{3}$

(b) $c_k = \frac{2k^2}{3k+1} = \frac{2k}{3 + 1/k}$

numerator $\to$ Large, denominator $\to 3$

Thus $c_k$ diverges

(c) $r_k = \left(\frac{3}{4}\right)^k$ since $\left|\frac{3}{4}\right| < 1$ $r_k \to 0$.

(d) $s_k = (-1)^k$ alternates between $-1$ and $1$;
thus $s_k$ diverges.

(e) Since natural log is an increasing function
$g_k = \ln(2k+1) \to$ does not converge.

(f) $f_k = \frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-(k)}{k(k+1)}$

$= \frac{1}{k(k+1)} \to 0$ as $k \to$ Large

(g) $l_k = \ln(k+1) - \ln(k) = \ln\left(\frac{k+1}{k}\right)$.

Since $\frac{k+1}{k} = \frac{1 + 1/k}{1} \to 1$ we have

$l_k \to \ln(1) = 0$.

③ (a) $2+4+\cdots+2014$

$$=2(1+2+\cdots+1007)$$

$$=2\,\frac{(1007)(1008)}{2} = (1007)(1008)$$

(b) $3+6+12+\cdots+768$

$$=3(1+2+4+\cdots+256)$$

$$=3\left(2^0+2^1+2^2+\cdots+2^8\right)$$

$$=3\,\frac{1-2^9}{1-2}$$

$$=3\,\frac{511}{1} = 1533$$

side

$$\begin{array}{r} 256 \\ 3\,\overline{)768} \\ 16 \\ \underline{15} \\ 18 \end{array}$$

(c) $1+8+27+\cdots+100^3$

$$=1^3+2^3+3^3+\cdots+100^3$$

$$=\left(\frac{100(101)}{2}\right)^2 = (50)^2(101)^2$$

(d)

$$\sum_{k=0}^{25}\left(\frac{4}{3}\right)^k = \frac{1-\left(\frac{4}{3}\right)^{26}}{1-\frac{4}{3}} = 3\left[\left(\frac{4}{3}\right)^{26}-1\right]$$

(e)

$$\sum_{k=1}^{10}(1+k^2) = \sum_{k=1}^{10}1 + \sum_{k=1}^{10}k^2$$

$$=10+\frac{(10)(11)(21)}{6}$$

③ cont.

(f) $\displaystyle\sum_{k=1}^{12} \frac{2\cdot 3^k}{5^{2k}} = 2\sum_{k=1}^{12}\left(\frac{3}{25}\right)^k$

change index
$l = k-1$
$k = l+1$

$= 2\displaystyle\sum_{l=0}^{11}\left(\frac{3}{25}\right)^{l+1}$

$= 2\left(\frac{3}{25}\right)\displaystyle\sum_{l=0}^{11}\left(\frac{3}{25}\right)^{l} = \frac{6}{25}\,\frac{1-\left(\frac{3}{25}\right)^{12}}{1-\left(\frac{3}{25}\right)}$

$= \dfrac{6}{25}\,\dfrac{25}{22}\left(1-\left(\frac{3}{25}\right)^{12}\right)$

$= \dfrac{3}{11}\left[1-\left(\frac{3}{25}\right)^{12}\right]$

(g) aside: $\dfrac{1}{k^2+4k} = \dfrac{1}{k(k+4)} = \dfrac{1/4}{k} + \dfrac{-1/4}{k+4}$

$\displaystyle\sum_{k=1}^{42}\frac{1}{k^2+4k} = \sum_{k=1}^{42}\frac{1}{4}\left(\frac{1}{k} - \frac{1}{k+4}\right)$

$= \dfrac{1}{4}\left[\dfrac{1}{1} + \dfrac{1}{2} + \dots + \dfrac{1}{42} - \dfrac{1}{5} - \dfrac{1}{6} - \dots - \dfrac{1}{46}\right]$

$= \dfrac{1}{4}\left[1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{43} - \dfrac{1}{44} - \dfrac{1}{45} - \dfrac{1}{46}\right]$

5 (a)

$$\sum_{k=0}^{\infty} (0.7)^k = \frac{1}{1-0.7} = \frac{1}{0.3} = \frac{10}{3}$$

(b)

$$\frac{1}{k^2+k} = \frac{1}{k(k+1)} = \frac{1}{k} + \frac{-1}{k+1}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \sum_{k=1}^{\infty} \left[ \frac{1}{k} - \frac{1}{k+1} \right]$$

$$= \left[ \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \cdots \right] = 1.$$

(c)

$$\sum_{k=1}^{\infty} \frac{2(-3)^k}{5^{2k}} = 2 \sum_{k=1}^{\infty} \left( \frac{-3}{25} \right)^k$$

$\ell = k-1$

$k = \ell+1$

$$= 2 \sum_{\ell=0}^{\infty} \left( \frac{-3}{25} \right)^{\ell+1}$$

$$= 2 \left( \frac{-3}{25} \right) \frac{1}{1 - \left( \frac{-3}{25} \right)}$$

$$= \frac{-6}{25} \cdot \frac{25}{28}$$

$$= -\frac{3}{14}$$

(8) (b)

(i) We ~~over~~ under estimate by

$$\lim_{N\to\infty} \int_0^N e^x \, dx = \lim_{N\to\infty} \left[ e^x \right]_0^N$$

$$= \lim_{N\to\infty} \left[ e^N - 1 \right] \quad \text{which diverges.}$$

Thus $\sum_{k=0}^{\infty} e^k$ diverges.

(ii) Look at integral

$$\lim_{N\to\infty} \int_0^N e^{-x} \, dx = \lim_{N\to\infty} \left[ -e^{-x} \right]_0^N$$

$$= \lim_{N\to\infty} \left[ -e^{-N} + 1 \right] = 1$$

Thus $\sum_{k=0}^{\infty} e^{-k}$ converges.

(iii) We compute $\lim_{N\to\infty} \int_1^N \frac{2}{\sqrt{x}} \, dx = \lim_{N\to\infty} \int_1^N 2x^{-1/2} \, dx$

$$= \lim_{N\to\infty} \left[ 4x^{1/2} \right]_1^N = \lim_{N\to\infty} \left[ 4\sqrt{N} - 4 \right] \to \infty$$

Thus series diverges.

8 (b) cont

(iv) Compute $\lim_{N\to\infty} \int_1^N \frac{2}{(\sqrt{x})^3}\,dx$

$$= \lim_{N\to\infty} \int_1^N 2x^{-3/2}\,dx$$

$$= \lim_{N\to\infty} \left[-4\,x^{-1/2}\right]_1^N$$

$$= \lim_{N\to\infty} \left[\frac{-4}{\sqrt{N}} - \frac{-4}{1}\right] = 4$$

Thus $\sum_{k=1}^{\infty} \frac{2}{(\sqrt{k})^3}$ converges

(v) compute $\lim_{N\to\infty} \int_0^N \frac{1}{1+x^2}\,dx$

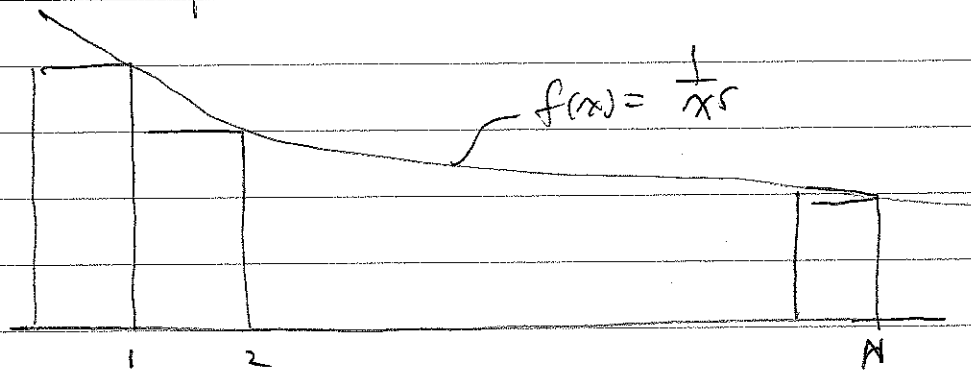
$$= \lim_{N\to\infty} \left[\tan^{-1}(x)\right]_0^N$$

$$= \lim_{N\to\infty} \left[\tan^{-1}(N) - 0\right] = \frac{\pi}{2}$$

Thus $\sum_{k=0}^{\infty} \frac{1}{1+k^2}$ converges.

(10) (b)

We draw picture




We **over** estimate $S_N$ by

$$S_N \le 1 + \int_1^N \frac{1}{x^5}\,dx$$

$$= 1 + \int_1^N x^{-5}\,dx$$

$$= 1 + \left[ -\frac{1}{4} x^{-4} \right]_1^N$$

$$= 1 + \left[ \frac{1}{4} - \frac{1}{4N^4} \right] \longrightarrow 1 + \frac{1}{4} \quad \text{as } N \to \infty$$

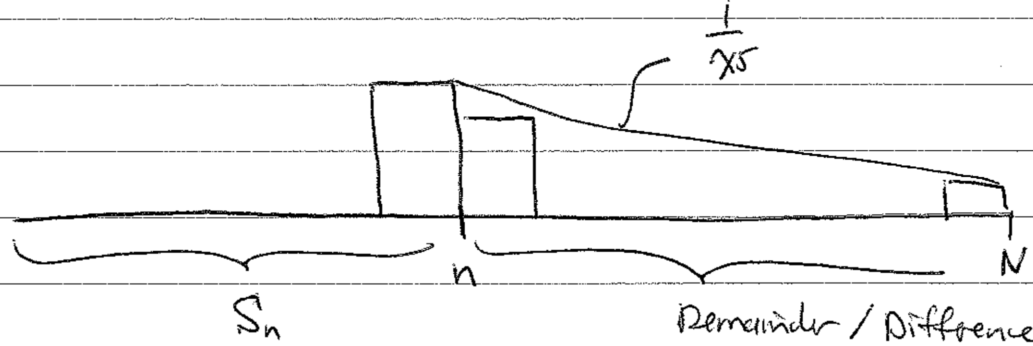
Thus $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^5} \le \frac{5}{4}$.

We **under** estimate by $\displaystyle\int_1^N \frac{1}{x^5}\,dx \longrightarrow \frac{1}{4}$

Thus

$$\frac{1}{4} \le \sum_{k=1}^{\infty} \frac{1}{k^5} \le \frac{5}{4}.$$

10 (d) We draw a picture




The diffrence is less than

$$\lim_{N\to\infty} \int_n^N \frac{1}{x^5}\,dx = \lim_{N\to\infty} \int_n^N x^{-5}\,dx$$

$$= \lim_{N\to\infty} \left[ \frac{-1}{4} x^{-4} \right]_n^N = \lim_{N\to\infty} \left[ \frac{-1}{4N^4} - \frac{-1}{4n^4} \right]$$

$$= \frac{1}{4n^4}$$

Thus we want $\frac{1}{4n^4} \leq 10^{-6}$

$$\frac{1}{4} 10^6 \leq n^4$$

$$\left(\frac{1}{4}\right)^{1/4} 10^{3/2} \leq n$$

We can choose, for example $n = 1000$, but can certainly do better...

(11) (b)

(ii) $\sum_{k=1}^{\infty} \frac{2}{3k+1}$ looks similar to $\sum_{k=1}^{\infty} \frac{(\#)}{k}$

Let's investigate:

$$\frac{2}{3k+1} \geq \frac{(\#)}{k}$$

$$2k \geq (\#)(3k+1)$$

$$[2-3(\#)]\,k \geq (\#)$$

$$\underbrace{\frac{2-3(\#)}{(\#)}}\, k \geq 1$$

would be great if this is $= 1$,

$$\frac{2-3(\#)}{(\#)} = 1 \iff 2-3(\#) = (\#)$$

$$2 = 4(\#)$$

$$\tfrac{1}{2} = (\#)$$

Thus whenever $k \geq 1$ we have

$$\frac{2}{3k+1} \geq \frac{1/2}{k}$$

Thus

$$\sum_{k=1}^{\infty} \frac{2}{3k+1} \geq \underbrace{\sum_{k=1}^{\infty} \frac{1/2}{k}}_{\text{this diverges}}$$

Thus $\sum_{k=1}^{\infty} \frac{2}{3k+1}$ diverges.

(11) (b) (ii) $\sum_{k=1}^{\infty} \frac{2}{k^3+1}$ looks like $\sum \frac{(\#)}{k^3}$.

In fact $\sum_{k=1}^{\infty} \frac{2}{k^3+1} \leq \sum_{k=1}^{\infty} \frac{2}{k^3}$

$= 2 \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^3}}_{\text{converges by p-series.}}$

Thus $\sum_{k=1}^{\infty} \frac{2}{k^3+1}$ converges.

(iii) $\sum_{k=1}^{\infty} \frac{1}{k^2+1} \leq \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{\text{converges by p-series}}$

Thus $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ converges.

(11) (b) (iv)

$$\sum_{k=1}^{\infty} \frac{2}{\sqrt{3k+1}} \quad \text{looks like} \quad \sum \frac{(\#)}{\sqrt{k}}$$

investigate:

want $$\frac{2}{\sqrt{3k+1}} \geq \frac{(\#)}{\sqrt{k}}$$

$$\frac{4}{3k+1} \geq \frac{(\#)^2}{k}$$

$$4k \geq (\#)^2 (3k+1)$$

$$\underbrace{\frac{[4-3(\#)^2]}{(\#)^2}}_{\text{want this } = 1} k \geq 1$$

$$4-3(\#)^2 = (\#)^2$$

$$4 = 4(\#)^2$$

$$\leadsto (\#) = 1.$$

Thus $$\sum_{k=1}^{\infty} \frac{2}{\sqrt{3k+1}} \geq \underbrace{\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}}_{\text{diverges by p-series}}.$$

Thus $$\sum_{k=1}^{\infty} \frac{2}{\sqrt{3k+1}} \text{ diverges.}$$

(13) (a) A series $\sum_{k=*}^{\infty} a_k$ converges absolutely if $\sum_{k=*}^{\infty} |a_k|$ also converges.

If $\sum_{k=*}^{\infty} a_k$ converges, but $\sum_{k=*}^{\infty} |a_k|$ does not, then $\sum_{k=*}^{\infty} a_k$ converges conditionally.

(b) The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges conditionally.

(c) The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges absolutely.

(d) (i) converges by alternating principle, but $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges by p-series.

Thus conditional convergence.

(ii) $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-series,

thus original series converges absolutely.

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by alternating series.

But $\displaystyle\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p series.

Thus original series converges conditionally.

(iv) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$ converges by alternating.

Look at $$\frac{\left|\frac{(-1)^{k+1}}{(k+1)!}\right|}{\left|\frac{(-1)^k}{k!}\right|} = \frac{k!}{(k+1)!} = \frac{1}{k+1} \to 0$$

Thus by ratio test we have absolute convergence.

(v) $\displaystyle\sum_{k=1}^{\infty} \left|\frac{\sin(k)}{k^5}\right| = \sum_{k=1}^{\infty} \frac{1}{k^5}$ which converges by p-series.

Thus $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{k^5}$ converges absolutely.

(vi) [use Ratio test $\to$ absolute convergence.]

(4) (a) The $n^{th}$ order Taylor polynomial for $f$ near $x_*$ is the polynomial

$$p_n(x) = a_0 + a_1(x - x_*) + a_2(x - x_*)^2 + \dots$$

$$\dots + a_n(x - x_*)^n$$

where

$$a_k = \frac{f^{(k)}(x_*)}{k!}$$

(b) We focus on $e^x$.

$$\frac{d^k}{dx^k}\left[e^x\right]_{x=0} = 1$$

Thus $$a_k = \frac{1}{k!}$$

and

$$p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$$

To approximate $e^{3x}$ we look at

$$p_3(3x) = 1 + 3x + \frac{1}{2}(3x)^2 + \frac{1}{3!}(3x)^3$$

$$= 1 + 3x + \frac{3^2}{2}x^2 + \frac{3^3}{3!}x^3$$

(c) We easily see that

$$f^{(k)}(5) = 4^k e^{20}$$

Thus $a_k = \frac{4^k e^{20}}{k!}$ and

$$e^{4x} \approx e^{20} + 4e^{20}(x-5) + \frac{4^2 e^{20}}{2}(x-5)^2$$

$$+ \frac{4^3 e^{20}}{6}(x-5)^3$$

(d) We compute

| | | |
|---|---|---|
| $f(x) = \sin(x)$ | $f(0) = 0$ | $a_0 = 0$ |
| $f'(x) = \cos(x)$ | $f'(0) = 1$ | $a_1 = 1$ |
| $f''(x) = -\sin(x)$ | $f''(0) = 0$ | $a_2 = 0$ |
| $f^{(3)}(x) = -\cos(x)$ | $f^{(3)}(0) = -1$ | $a_3 = \frac{-1}{3!}$ |
| $f^{(4)}(x) = \sin(x)$ | $f^{(4)}(0) = 0$ | $a_4 = 0$ |
| $f^{(5)}(x) = \cos(x)$ | $f^{(5)}(0) = 1$ | $a_5 = \frac{1}{5!}$ |

Thus $\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

(e) Similar process yields

$$\cos(x) \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

(f) Compute

$f(x) = \ln(x)$ $\qquad f(1) = 0$

$f'(x) = x^{-1}$ $\qquad f'(1) = 1$

$f''(x) = -x^{-2}$ $\qquad f''(1) = -1$

$f'''(x) = (-1)(-2)x^{-3}$ $\qquad f^{(3)}(1) = 2$

$$\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3$$

(g) Compute

$f(x) = \ln(1+x)$ $\qquad f(0) = 0$ $\qquad a_0 = 0$

$f'(x) = (1+x)^{-1}$ $\qquad f'(0) = 1$ $\qquad a_1 = 1$

$f''(x) = -(1+x)^{-2}$ $\qquad f''(0) = -1$ $\qquad a_2 = \frac{-1}{2}$

$f^{(3)}(x) = 2(1+x)^{-3}$ $\qquad f'''(0) = 2$ $\qquad a_3 = \frac{2}{6}$

$$\ln(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$