Finite Differences and Solvability of Differential Equation Problems

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Problems?

Does there exist a function $u\colon [0,1] \to \mathbb{R}$ such that

- $u''(t) + \sqrt{17}\, u(t) = \cos(3t)$,
- $u(0) = 0$, and
- $u'(0) = 1$?

What about a function $u\colon [0,1] \to \mathbb{R}$ such that

- $u''(t) + 42\pi\, u(t) = e^{-t}$,
- $u(0) = 0$, and
- $u(1) = 0$?

Why?

- Science...
- Idle curiosity...
- Learn something!

General problem

Find function $u\colon [0,1] \to \mathbb{R}$ satisfying

- $u''(t) + \lambda\, u(t) = f(t)$ and
- Conditions at $t = 0$ and/or $t = 1$.

Special case

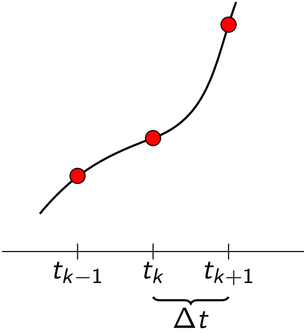
Special case: $\lambda = 0$

- Integrate twice. . . two free constants. . .
- Need two conditions

TLAs:

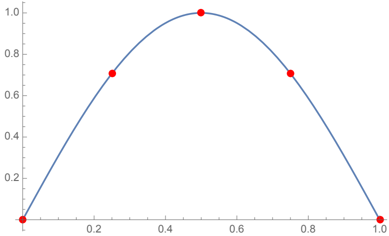
- IVP $\rightsquigarrow$ Cauchy: $u(0) = u_0$, $u'(0) = v_0$
- BVP $\rightsquigarrow$ Dirichlet: $u(0) = 0$, $u(1) = 0$

Approximating the problem. . .



- Divide domain $[0, 1]$ in to intervals of size $\Delta t = \frac{1}{n+1}$
- Only record values at $t_0 = 0, t_1 = \Delta t, t_2 = 2(\Delta t), \ldots, t_{n+1} = 1$
- $u(t) \leftrightarrow \mathbf{u} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} u(t_0) \\ u(t_1) \\ \vdots \\ u(t_{n+1}) \end{pmatrix}$

Example ($n = 3$)



$$u(t) = \sin(\pi t) \qquad \mathbf{u} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

What about derivatives?

- Basic approximation

$$u'(t_k) = \frac{1}{\Delta t}\left[u_{k+1} - u_k\right] + \mathcal{O}(\Delta t)$$

- Use Taylor for better approximations

- Taylor:

$$u(t_{k+1}) = u(t_k) + u'(t_k)\,\Delta t + \frac{u''(t_k)}{2}\,(\Delta t)^2 + \frac{u'''(t_k)}{6}\,(\Delta t)^3 + \mathcal{O}\left((\Delta t)^4\right)$$

$$u(t_{k-1}) = u(t_k) - u'(t_k)\,\Delta t + \frac{u''(t_k)}{2}\,(\Delta t)^2 - \frac{u'''(t_k)}{6}\,(\Delta t)^3 + \mathcal{O}\left((\Delta t)^4\right)$$

- Derivatives modulo $\mathcal{O}\left((\Delta t)^2\right)$

$$u''(t_k) \approx \frac{1}{(\Delta t)^2}\left[u_{k-1} - 2u_k + u_{k+1}\right]$$

$$u'(t_k) \approx \frac{1}{2(\Delta t)}\left[u_{k+1} - u_{k-1}\right]$$

Constructing approximate problems

	DE problem	Approximate problem
Var.	function $u(t)$	vector $\mathbf{u} = (u_k)$
Eqn.	$u'' + \lambda u = f$	$\dfrac{u_{k-1} - 2u_k + u_{k+1}}{(\Delta t)^2} + \lambda u_k = f_k$
IC	$u(0) = u_0,\ u'(0) = v_0$	$u_0 = u_0,\ u_1 = u_0 + v_0(\Delta t)$
BC	$u(0) = 0,\ u(1) = 0$	$u_0 = 0,\ u_{n+1} = 0$

- Approximate problem is *algebraic*
- Equations are *linear*

Approximate initial value problem

- Initial conditions $\rightsquigarrow u_0, u_1$
- Approximate equation

$$u_{k+1} = [2 - \lambda(\Delta t)^2]u_k - u_{k-1} + f_k$$

  determines remaining entries in $\mathbf{u}$.
- All initial conditions lead to solutions
- Parameter λ does not affect solvability
- Peano, 1890: Same is true for original IVP
  - Approximate solution $\rightsquigarrow$ actual solution
  - Interesting historical notes...

Théorème.

9. $f \,\varepsilon\, (\mathrm{Kq_n})/\mathrm{Q} \therefore h, k \,\varepsilon\, \mathrm{Q} \,.\, h < k : \supset_{h,k} .\, fh \supset fk \therefore p \,\varepsilon\, \mathrm{Q} : h \,\varepsilon\, \mathrm{Q} \,.\supset_h$
$.\, fh \cap \theta \overline{\mathrm{m}} p -\!= \Lambda :: \supset .\, f0 -\!= \Lambda.$

Démonstration.

(1) $\mathrm{Hp} \,.\, c \,\varepsilon\, \mathrm{Kq_n} \,.\, h \,\varepsilon\, \mathrm{Q} \,.\, c \cap fh = \Lambda \,.\, k \,\varepsilon\, \theta h : \supset .\, c \cap fk = \Lambda.$

(2) $\mathrm{Hp} \,.\, c, c' \,\varepsilon\, \mathrm{Kq_n} \,.\, h, h' \,\varepsilon\, \mathrm{Q} \,.\, c \cap fh = \Lambda \,.\, c' \cap fh' = \Lambda \,.\, h'' \,\varepsilon\, \mathrm{Q} \cap \theta h$
$\cap\, \theta h' : \supset .\, (c \cup c') \cap fh'' = \Lambda.$ $\{(1) \supset (2)\}.$

(3) $\mathrm{Hp} \,.\, c, c' \,\varepsilon\, \mathrm{Kq_n} : \supset \therefore\!\cdot\, h \,\varepsilon\, \mathrm{Q} \,.\, c \cap fh = \Lambda : -\!=_h \Lambda \therefore h' \,\varepsilon\, \mathrm{Q} \,.\, c'$
$\cap fh' = \Lambda : -\!=_{h'} \Lambda :: = \therefore h'' \,\varepsilon\, \mathrm{Q} \,.\, (c \cup c') \cap fh'' = \Lambda : -\!=_{h''} \Lambda.$
$\{(2) \supset (3)\}.$

(4) $\mathrm{Hp} \,.\, u = \mathrm{Kq_n} \cap \overline{c\varepsilon}\, [h \,\varepsilon\, \mathrm{Q} \,.\, c \cap fh = \Lambda : =_h \Lambda] \,.\, (3) : \supset \therefore\!\cdot\, u$
$\varepsilon\, \mathrm{KKq_n} :: c, c' \,\varepsilon\, \mathrm{Kq_n} \,.\supset_{c,c'} \therefore c \cup c' \,\varepsilon\, u \,.\, = : c \,\varepsilon\, u \,.\cup.\, c'$
$\varepsilon\, u :: \theta \overline{\mathrm{m}} p \,\varepsilon\, u.$

(5) $u \,\varepsilon\, \mathrm{KKq_n} :: c, c' \,\varepsilon\, \mathrm{Kq_n} \,.\supset_{c,c'} \therefore c \cup c' \,\varepsilon\, u \,.\, = : c \,\varepsilon\, u \,.\cup.\, c' \,\varepsilon\, u$
$:: s \,\varepsilon\, u \,.\, \mathrm{l'ms} \,\varepsilon\, \mathrm{q} \therefore\!\cdot \supset :: x \,\varepsilon\, \mathrm{C}s : k \,\varepsilon\, \mathrm{Q} \,.\supset_k .\, x + \theta \overline{\mathrm{m}} k \,\varepsilon\, u$
$\therefore -\!=_x \Lambda.$

{Cantor, *Ueber unendliche, lineare Punktmannichfaltigkeiten*, Math. Ann. XXIII, p. 454}.

(6) $\mathrm{Hp} \,.\, u = \mathrm{Kq_n} \cap \overline{c\varepsilon}\, [h \,\varepsilon\, \mathrm{Q} \,.\supset_h .\, c \cap fh -\!= \Lambda] \,.\, (4) \,.\, (5) : \supset :: x$
$\varepsilon\, \theta \overline{\mathrm{m}} p : k \,\varepsilon\, \mathrm{Q} \,.\supset_k .\, x + \theta \overline{\mathrm{m}}\, k \,\varepsilon\, u \therefore -\!=_x \Lambda.$

(7) $\mathrm{Hp} \,.\, (6) : \supset \therefore\!\cdot\, x \,\varepsilon\, \theta \overline{\mathrm{m}} p \therefore k \,\varepsilon\, \mathrm{Q} \,.\supset_k : h \,\varepsilon\, \mathrm{Q} \,.\supset_h .\, (x + \theta \overline{\mathrm{m}} k) \cap fh$
$-\!= \Lambda :: -\!=_x \Lambda.$

(8) $\mathrm{Hp} .(7) : \supset :: x \,\varepsilon\, \theta \overline{\mathrm{m}} p : h \,\varepsilon\, \mathrm{Q} .\supset_h .\, \mathrm{l_1 m}\, (fh - x) = 0 \therefore -\!=_x \Lambda.$

(9) $\mathrm{Hp} \,.\, (8) : \supset : x \,\varepsilon\, f0 \,.\, -\!=_x \Lambda.$

$\mathrm{Hp} \,.\, (9) : \supset .\, \mathrm{Ts}.$

A boundary value problem

$$u''(t) + \lambda u(t) = \sin(\pi t),$$
$$u(0) = 0, \qquad u(1) = 0$$

- $\lambda \neq \pi^2 \rightsquigarrow$ General solution is

$$u(t) = A \sin(\sqrt{\lambda}\, t) + B \cos(\sqrt{\lambda}\, t) + \frac{1}{\lambda - \pi^2} \sin(\pi t)$$

- $\lambda = \pi^2 \rightsquigarrow$ General solution is

$$u(t) = A \sin(\sqrt{\lambda}\, t) + B \cos(\sqrt{\lambda}\, t) - \frac{t}{2\pi} \cos(\pi t)$$

Conclude

- $\lambda = \pi \rightsquigarrow$ unique solution
- $\lambda = \pi^2 \rightsquigarrow$ no solution
- $\lambda = 4\pi^2 \rightsquigarrow$ infinite number of solutions

Huh?

Solvability of BVP seems very delicate. . .

Plan:

- Study approximate problem
- Try to gain insight about when solutions exist

Method:

- Use tools for studying linear, algebraic equations

Approximate equation $\mathbf{Mu} = \mathbf{f}$

Approximate

$$u''(t) + \lambda u(t) = f(t)$$

by

$$\frac{1}{(\Delta tx)^2}\left[u_{k-1} - 2u_k + u_{k+1}\right] + \lambda u_k = f_k, \quad k = 1, \ldots, n.$$

Using $u_0 = 0$ and $u_{n+1} = 0$ write

$$\underbrace{\begin{bmatrix} \alpha & 1 & & & \\ 1 & \alpha & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha & 1 \\ & & & 1 & \alpha \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix}}_{\mathbf{u}} = \underbrace{(\Delta t)^2 \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}}_{\mathbf{f}}$$

Here $\alpha = -2 + \lambda(\Delta t)^2$

Example when $n = 3$

$$u''(t) + \lambda u(t) = \sin(\pi t)$$
$$u(0) = 0 \qquad u(1) = 0$$

approximated by

$$\begin{bmatrix} -2 + \frac{1}{16}\lambda & 1 & 0 \\ 1 & -2 + \frac{1}{16}\lambda & 1 \\ 0 & 1 & -2 + \frac{1}{16}\lambda \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

- First/last entry = 0 is “built in”

Some linear algebra

Proposition

For linear transformation $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^n$, *the solvability of the equation*

$$(\bigstar) \qquad \mathcal{L}(\mathbf{u}) = \mathbf{f}$$

is described as follows:

1. *If* $\ker(\mathcal{L}) = \{\mathbf{0}\}$, *then* $(\bigstar)$ *has a unique solution for all* $\mathbf{f} \in \mathbb{R}^n$.
2. *If* $\ker(\mathcal{L}) \neq \{\mathbf{0}\}$, *then either*
   - 2.1 $\mathbf{f} \in \ker(\mathcal{L}^*)^\perp$, *in which case* $(\bigstar)$ *has multiple solutions, or*
   - 2.2 $\mathbf{f} \notin \ker(\mathcal{L}^*)^\perp$, *in which case* $(\bigstar)$ *does not have a solution.*

Here $\mathcal{L}^*$ is the adjoint transformation of $\mathcal{L}$:
If $\mathcal{L}(\mathbf{u}) = \mathbf{M}\mathbf{u}$, then $\mathcal{L}^*(\mathbf{u}) = \mathbf{M}^t\mathbf{u}$

Example when $n = 3$

$$\begin{bmatrix} -2+\frac{1}{16}\lambda & 1 & 0 \\ 1 & -2+\frac{1}{16}\lambda & 1 \\ 0 & 1 & -2+\frac{1}{16}\lambda \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Compute

$$\det \mathbf{M} = \left(\frac{\lambda}{16} - 2\right)\left(\frac{\lambda}{16} - 2 + \sqrt{2}\right)\left(\frac{\lambda}{16} - 2 - \sqrt{2}\right).$$

- Unique solution if $\lambda \notin \{32, 32 - 16\sqrt{2}, 32 + 16\sqrt{2}\}$
- If $\lambda = 32$ then $\text{null}\,(\mathbf{M}) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$

  Since $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$ we have multiple solutions.
- Also multiple solutions if $\lambda = 32 + 16\sqrt{2}$.

Example when $n = 3$ continued

$$\begin{bmatrix} -2+\frac{1}{16}\lambda & 1 & 0 \\ 1 & -2+\frac{1}{16}\lambda & 1 \\ 0 & 1 & -2+\frac{1}{16}\lambda \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

- If $\lambda = 32 - 16\sqrt{2}$ then $\text{null}\,(\mathbf{M}) = \text{span}\left\{\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\right\}$
- Since $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \neq 0$, no solutions.

Summary of example when $n = 3$

- We understand solvability of approximate system by computing $\det \mathbf{M}$
- For most values of λ, there exists unique solution
- For exceptional values of λ either no solution or many solutions, depending on whether RHS is orthogonal to null space.

Now we need to do this for general $n \times n$ matrix $\mathbf{M}$...

Computing determinants

- Let $d_n = \begin{vmatrix} \alpha & 1 & & & \\ 1 & \alpha & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha & 1 \\ & & & 1 & \alpha \end{vmatrix}$
- Expand around first row: $d_n = \alpha d_{n-1} - d_{n-2}$
- Subject to $d_1 = \alpha$, $d_2 = \alpha^2 - 1$
- When $\alpha^2 \neq 4$ we find $d_n = \frac{1}{\sqrt{\alpha^2-4}}\left[(z_+)^{n+1} - (z_-)^{n+1}\right]$, where
$$z_\pm = \frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - 1}$$
- When $\alpha^2 = 4$, $d_n \neq 0$.

Exceptional values of λ

- $\det \mathbf{M} = 0$ when

$$\lambda_l = 2(n+1)^2 \left[1 - \cos\left(\frac{\pi l}{n+1}\right)\right], \quad l = 1, 2, \ldots, n.$$

- With $\lambda = \lambda_l$

$$\text{null}\,(\mathbf{M}) = \text{span}\left\{ \mathbf{v}_l = \begin{pmatrix} \sin\left(\frac{\pi l}{n+1}\right) \\ \sin\left(\frac{2\pi l}{n+1}\right) \\ \vdots \\ \sin\left(\frac{n\pi l}{n+1}\right) \end{pmatrix} \right\}$$

- This looks familiar!

Solvability of approximate problem

Translating the linear algebra theorem, we have

1. If λ is not one of the exceptional values, then the null space of $\mathbf{M}$ is trivial and the approximate problem has a unique solution for each vector $\mathbf{f}$.
2. If $\lambda = \lambda_l$, one of the exceptional values, then the null space of $\mathbf{M}$ is non-trivial and is spanned by $\mathbf{v}_l$. Furthermore,
   2.1 if $\mathbf{v}_l \cdot \mathbf{f} = 0$ the approximate problem has multiple solutions, while
   2.2 if $\mathbf{v}_l \cdot \mathbf{f} \neq 0$ the approximate problem does not admit any solutions.

This sets expectations for original problem. . .

Returning to original BVP

- Expectation: Original problem modeled by limit $n \to \infty$
- Compute using Taylor expansion of cosine

$$\lambda_l = 2(n+1)^2 \left[1 - \left\{ 1 - \frac{1}{2} \left(\frac{\pi l}{n+1} \right)^2 + \mathcal{O}\left(n^{-4}\right) \right\} \right]$$
$$= (\pi l)^2 + \mathcal{O}\left(n^{-2}\right)$$

- Conjecture:
  - If $\lambda \neq (\pi l)^2$ for $l = 0, 1, 2, \ldots$ then there exists unique solution.
  - If $\lambda = (\pi l)^2$ for some $l = 0, 1, 2, \ldots$ then either no solution or many solutions...
    ...depending on "orthogonality to $v_l(t) = \sin(\pi l t)$"

Orthogonality

What should "orthogonality to $\sin(\pi l t)$ mean?

- In approximate setting, it means

$$0 = \begin{pmatrix} f\left(\frac{1}{n+1}\right) \\ f\left(\frac{2}{n+1}\right) \\ \vdots \\ f\left(\frac{n}{n+1}\right) \end{pmatrix} \cdot \begin{pmatrix} \sin\left(\frac{\pi l}{n+1}\right) \\ \sin\left(\frac{2\pi l}{n+1}\right) \\ \vdots \\ \sin\left(\frac{n\pi l}{n+1}\right) \end{pmatrix} = \sum_{k=0}^{n} f(t_k)\sin(\pi l t_k)$$

- Riemann sum for $\int_0^t f(t)v_l(t)\,dt, \qquad v_l(t) = \sin(\pi l t)$
- Use inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$

Boundary value problem, revisited

$$
\begin{aligned}
u''(x) + \lambda u(x) &= \sin(\pi x), & x \in (0,1) \\
u(0) = 0, \qquad u(1) &= 0, & \lambda > 0
\end{aligned}
$$

Are there solutions?

- $\lambda = \pi$ not an exceptional value
  $\rightsquigarrow$ unique solution
- $\lambda = \pi^2 = \lambda_1$
  Since $f(t) = v_1(t) \sin(\pi x)$ $\rightsquigarrow$ no solution
- $\lambda = 4\pi^2 = \lambda_2$
  Is $f(t) = \sin(\pi t)$ orthogonal to $v_2(t) = \sin(2\pi x)$?
  Yes $\rightsquigarrow$ multiple solutions

More formally. . .

- View $\mathcal{L} = \frac{d^2}{dt^2} - \lambda$ as linear transformation $V_0 \to V$
  - $V = \{u : [0, 1] \to \mathbb{R} \mid \text{ finite norm } \}$
  - $V_0 = \{u \in V \mid u(0) = 0, u(1) = 0\}$
- Express generic BVP as finding u such that $\mathcal{L}(u) = f$
- "Same theorem" describes solvability:
  1. If $\ker(\mathcal{L})$ contains only the zero function, then we have a unique solution for all f.
  2. If $\ker(\mathcal{L})$ contains a non-zero function, then either
     - 2.1 $f \in \ker(\mathcal{L}^*)^\perp$, in which case we have multiple solutions, or
     - 2.2 $f \notin \ker(\mathcal{L}^*)^\perp$, in which case we have has no solution.
- Case 1 occurs for all but a countable list of λ

"Fredholm alternative". . . "spectrum". . . "functional analysis". . .

Summary & Conclusion

- Numerical analysis sheds interesting light on solvability of problems.
- Existence and uniqueness theory for boundary value problems is "complicated" $\leftrightarrow$ "interesting"
- One can construct an approximate problem involving linear mappings of $\mathbb{R}^n$
- Theory for mappings of $\mathbb{R}^n$ has same structure theory for linear mappings of function spaces.
- Vector spaces are everywhere!

Thank you!