Weakly asymptotically hyperbolic solutions to the Einstein constraint equations

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University of Connecticut, March 2016

Abstract

We introduce a class of "weakly asymptotically hyperbolic" geometries whose sectional curvatures tend to -1 and are C^0 but not necessarily C^2 conformally compact. We establish Fredholm results for geometric elliptic operators in this setting.

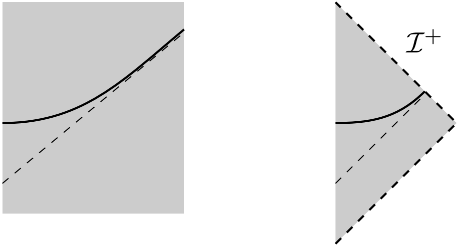
We use these results to construct constant-mean-curvature solutions to the Einstein constraint equations in the weakly asymptotically hyperbolic setting. We furthermore can ensure that our solutions satisfy the shear-free condition, which is necessary for any spacetime development to admit a regular conformal boundary at future null infinity.

- joint with James Isenberg, John M. Lee, Iva Stavrov Allen
- arXiv: `1506.03399`, `1506.06090`; to appear in CAG, CQG

Motivation

Minkowski spacetime $(\mathbb{R}^4, -dt^2 + dx^2 + dy^2 + dz^2)$

- Penrose diagram of compactification:



- Foliation by spacelike hyperboloids

Asymptotically flat spacetimes

- Conformal structure asymptotically Minkowskian
- "Hyperboloidal" foliations

Goal

Construct initial data (g, K) on M such that:

- (CMC) extrinsic curvature $K = -g + \Sigma$, where $\operatorname{tr}_g \Sigma = 0$
- (AH) g is "asymptotically hyperbolic" and Σ decays
- (C) Einstein constraint equations

$$\mathrm{R}[g] - |\Sigma|_g^2 + 6 = 0 \qquad \operatorname{div}_g \Sigma = 0$$

- (SF) "Shear-free condition" holds

(SF) necessary for C^0 conformal compactification

What are the issues?

1. Put (SF) in to conformal method:
   - Want $g = \phi^4\lambda$ and $\Sigma = \phi^{-2}(\mu + \mathcal{D}_\lambda W)$ to solve (C)
   - Requires

$$\mathcal{D}_\lambda^* \mathcal{D} W = -\operatorname{div}_\lambda \mu$$
$$\Delta_\lambda \phi = \frac{1}{8}\,\mathrm{R}[\lambda]\phi - \frac{1}{8}|\mu + \mathcal{D}_\lambda W|_\lambda^2 \phi^{-7} + \frac{3}{4}\phi^5$$

     where $\mathcal{D}_\lambda W = \frac{1}{2}\mathcal{L}_W \lambda - \frac{1}{3}(\operatorname{div}_\lambda W)\lambda$
   - Need a conformally invariant expression of (SF)

2. Regularity at the conformal boundary. We want a regularity class of metrics that is…
   - strong enough to obtain elliptic theory, and to define shear-free condition, but…
   - weak enough that the problem "closes"

Strongly asymptotically hyperbolic metrics

- Setup: $M = \mathrm{int}(\overline{M})$; $\rho \in C^\infty(\overline{M})$ is a defining function:

$$\rho \geq 0 \qquad \rho^{-1}(0) = \partial M \qquad d\rho \neq 0 \text{ along } \partial M.$$

- metric g is (strongly) C^k asymptotically hyperbolic if

$$\overline{g} := \rho^2 g \in C^k(\overline{M}) \qquad \text{conformally compact}$$
$$|d\rho|_{\overline{g}} \to 1 \text{ at } \partial M, \qquad \text{and thus } \operatorname{Riem}[g] \to -\operatorname{id}$$

- all previous work assumes $k \geq 2$

Previous results: Yamabe problem

Andersson, Chruściel, Friedrich: Yamabe problem ($\Sigma \equiv 0$):

- Suppose λ is C^∞ asymptotically hyperbolic
- $g = \phi^4 \lambda$ may not be C^∞ conformally compact
- Rather $\overline{g} = \rho^2 g$ has "polyhomogeneous expansion" at $\rho = 0$, polynomial in ρ^s and $(\log \rho)^k$
- $\overline{g} \in C^\infty(\overline{M})$ iff shear-free condition holds:

$$\rho \Sigma = \text{traceless } \operatorname{Hess}_{\overline{g}} \rho \qquad \text{along } \partial M,$$

in particular, if traceless Hessian vanishes at boundary.

Previous results: Shear-free condition

1. Andersson, Chruściel:
   - Generalize previous result to general μ
   - Many regularity classes; all require $C^{k\geq 0}$ conformal compactification
   - If $\overline{g} \in C^\infty(\overline{M})$, then log terms in expansion at boundary appear generically in $C^\infty(\overline{M})$ topology
2. Andersson, Chruściel:
   - Systematic investigation of log terms; resonance
   - Shear-free condition necessary for spacetime development to admit conformal boundary at future null infinity

Lessons:

- *Take shear-free condition seriously*
- *need suitable regularity class*

Some other previous results

1. Sakovich
   - "much better" spaces; insufficient for shear-free condition
2. A— , Stavrov Allen
   - "Smoothly compactifiable shear-free hyperboloidal data is dense in the physical topology"
   - Physical topology does not see conformal boundary
   - The "usual" weighted spaces don't improve situation:
     - either (SF) not defined,
     - or conformal structure can't vary.
3. Bahuaud, Gicquaud
   - Boundary regularity for Poincaré–Einstein metrics

Regularity on M and $\overline{M}$

Fix metric $\overline{h} \in C^\infty(\overline{M})$; associated AH metric $h = \rho^{-2}\overline{h}$ on M

- For covariant 2-tensors: $|u_{\cdot\cdot}|_{\overline{h}} = \rho^{-2}|u_{\cdot\cdot}|_h$ thus

$$u \in L^\infty(\overline{M}) \quad \leftrightarrow \quad u \in L^\infty_2(M) = \rho^2\, L^\infty(M)$$

- $u \in C^k(\overline{M})$ if $\mathcal{L}_{\overline{X}_1} \dots \mathcal{L}_{\overline{X}_l} u \in C^0(\overline{M})$ for $l \le k, \quad |\overline{X}_j|_{\overline{h}} \lesssim 1$
- $u \in C^k(M)$ if $\mathcal{L}_{X_1} \dots \mathcal{L}_{X_l} u \in C^0(M)$ for $l \le k, \quad |X_j|_h \lesssim 1$

Hybrid spaces: $u \in \mathscr{C}^{k,\alpha;m}(M)$ if

$$\underbrace{\mathcal{L}_{X_1} \dots \mathcal{L}_{X_p}}_{p \le k-m} \underbrace{\mathcal{L}_{\overline{X}_1} \dots \mathcal{L}_{\overline{X}_q}}_{q \le m} u \in C^{0,\alpha}_2(M)$$

Note: These are not *the V_b spaces of Melrose–Mazzeo*

Weakly asymptotically hyperbolic metrics

Idea

- "high" interior regularity for elliptic theory
- "low" boundary regularity

Details: with $\overline{g} = \rho^2 g$, we require

- Hybrid regularity

$$\begin{aligned} \overline{g} \in C_2^{k,\alpha}(M) \qquad & \longrightarrow \overline{g} \in L^\infty(\overline{M}) \\ \mathcal{L}_{\overline{X}}\overline{g} \in C_2^{k-1,\alpha}(M) \qquad & \longrightarrow \overline{g} \in W^{1,\infty}(\overline{M}) \subset C^{0,1}(\overline{M}) \end{aligned}$$

- Asymptotic negative curvature

$$\operatorname{Riem}[g] \to -\operatorname{id} \quad \leftrightarrow \quad |d\rho|_{\overline{g}} \to 1$$

We call g *weakly asymptotically hyperbolic* of class $\mathscr{C}^{k,\alpha;1}$.

- such regularity is sufficient for elliptic theory
- ...but $\operatorname{Hess}_{\overline{g}}(\rho) \in L^\infty(\overline{M})$ insufficient for shear-free condition

Regularity for shear-free condition

In order to accomodate shear-free condition we may also require

$$\mathcal{L}_{\overline{X}_1}\mathcal{L}_{\overline{X}_2}\overline{g} \in C_2^{k-2,\alpha}(M) \quad \longrightarrow \overline{g} \in W^{2,\infty}(\overline{M}) \subset C^{1,1}(\overline{M})$$

in which case we say g is weakly asymptotically hyperbolic of class $\mathscr{C}^{k,\alpha;2}$

- In general, we may define weakly asymptotically hyperbolic of class $\mathscr{C}^{k,\alpha;m}$ for $1 \leq m \leq k$.
- Case $2 \leq m = k$ was previously studied.

Elliptic theory

Previous work in asymptotically hyperbolic setting

- Mazzeo's edge calculus
- geometric approach: Lee; Andersson
- all require at least C^2 conformal compactification

We adapt the geometric approach of Lee

- Let g weakly AH of class $\mathscr{C}^{k,\alpha;m}$, $1 \le m$
- P is a second-order geometric elliptic operator arising from g then

$$P\colon C^{k,\alpha}_\delta(M) \to C^{k-2,\alpha}_\delta(M)$$
$$P\colon W^{k,p}_\delta(M) \to W^{k-2,p}_\delta(M)$$

are Fredholm of index zero for all δ in "Fredholm range"

Elliptic theory, continued

Key ideas of proof

- Fredholm range determined by model operator $\breve{P}$, corresponding to hyperbolic metric.
- Interior $C^{k,\alpha}$ regularity $\longrightarrow$ elliptic estimates, etc.
- Boundary $C^{0,1}$ regularity $\longrightarrow$ parametrix construction

Application to constraint equations

- We can solve elliptic equations in conformal method
- But: If g is weakly AH, is $\phi^4 g$ also weakly AH?

Boundary regularity for Yamabe case

Solve $\Delta_g \phi = \frac{1}{8}\,\mathrm{R}[g]\phi + \frac{3}{4}\phi^5$ where g is AH of class $\mathscr{C}^{k,\alpha;m}$.

For $\phi^4 g$ to be AH of class $\mathscr{C}^{k,\alpha;m}$ we need $\phi = 1 + \mathcal{O}(\rho^m)$

Write $\phi = 1 + u$ and use

- $\mathrm{R}[g] = -6 + \mathcal{O}(\rho)$
- $\Delta u = \rho^2 \partial_\rho^2 u - 2\rho\partial_\rho u + \rho^2 \not{\Delta} u$

to obtain

$$(\rho\partial_\rho + 1)(\rho\partial_\rho - 3)u = [\mathcal{O}(\rho)] + Q(u).$$

- Resonances are at ρ^3 and ρ^{-1} $\leftrightarrow$ Fredholm range
- Theory closes when $m = 1$.
- Need to work harder for $m = 2$... which we need for (SF)!

Boundary regularity for $m = 2$

Idea: Conformally deform metric to achieve $\mathrm{R}[g] = -6 + \mathcal{O}(\rho^2)$

- In C^∞ conformally compact case, accomplish this by ODE argument (Andersson, Chruściel, Friedrich)
- In weakly AH setting use more delicate regularization… group convolution on $\mathbb{H}$

Rely on conformal covariance of Yamabe/Lichnerowicz equation

Thus in weakly asymptotically hyperbolic setting

- we have elliptic theory… and regularity closes in $\mathscr{C}^{k,\alpha;m}$.

Only piece remaining is shear-free condition

Shear-free condition

In a conformally covariant manner, we want to enforce

$$\rho\Sigma = \text{traceless }\ \operatorname{Hess}_{\overline{g}}\rho \quad \text{along } \partial M$$

Define traceless tensor $\mathcal{H}_{\overline{g}}(\rho)$ to be

$$|d\rho|^6_{\overline{g}}\,\underbrace{\mathcal{D}_{\overline{g}}(|d\rho|^{-2}_{\overline{g}}\operatorname{grad}_{\overline{g}}\rho)} + \tfrac{1}{2}|d\rho|_{\overline{g}}\,\underbrace{\operatorname{div}_{\overline{g}}(|d\rho|_{\overline{g}}\operatorname{grad}_{\overline{g}}\rho)}\,(d\rho\otimes d\rho - \tfrac{1}{3}|d\rho|^2_{\overline{g}}\overline{g})$$

- Conformally covariant $\mathcal{H}_{\theta\overline{g}}(\rho) = \theta^{-2}\mathcal{H}_{\overline{g}}(\rho)$
- If g is AH of class $\mathscr{C}^{k,\alpha;2}$ then

$$\mathcal{H}_{\overline{g}}(\rho) = \text{traceless }\ \operatorname{Hess}_{\overline{g}}\rho \quad \text{along } \partial M$$

We can build (SF) in to conformal method by requiring $\mathcal{H}_{\overline{g}}(\rho)$ to be leading order term.

CMC AH SF constraints

Given metric λ, weakly AH of class $\mathscr{C}^{k,\alpha;m}$, and tensor μ we find

$$g = \phi^4 \lambda \quad \text{and} \quad K = -g + \phi^2 \rho^{-1} \mathcal{H}_{\overline{g}}(\rho) \phi^{-2} (\mu + \mathscr{D}_\lambda W)$$

satisfying constraint equations.

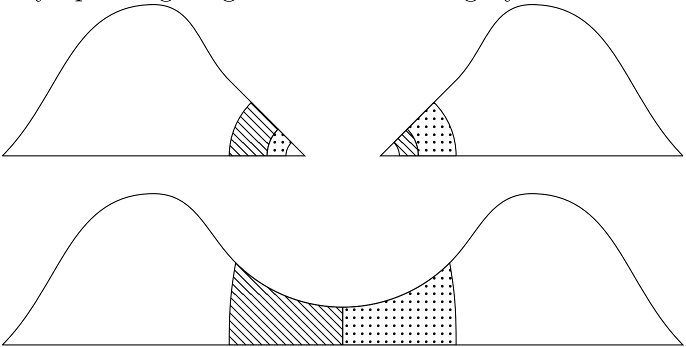
- For $m = 1, 2$ we have g weakly AH of class $\mathscr{C}^{k,\alpha;m}$
- For $m = 2$ the shear-free condition holds

Furthemore

- $(\lambda, \mu) \mapsto (g, K)$ is a continuous projection
- results in smooth and polyhomogeneous categories

Works in progress

- Asymptotic gluing in shear-free category



- Elliptic theory for operators arising from rough (Sobolev) weakly asymptotically hyperbolic metrics

Thank you!