Elliptic problems and weakly asymptotically hyperbolic manifolds

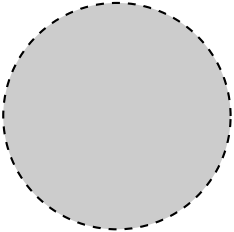
Paul T. Allen
joint work with
James Isenberg, John M. Lee and Iva Stavrov Allen
arXiv:1506.03399

January 2016

Part 1:
Asymptotically hyperbolic manifolds
and
the Yamabe problem

Models of hyperbolic space $(\mathbb{H}, \breve{g})$

Poincaré ball model



$$\breve{g} = \breve{\rho}^{-2} g_{\mathrm{Euclidean}}$$

$$\breve{\rho} = \tfrac{1}{2}(1 - |x|^2)$$

Half space model

$$\breve{g} = y^{-2} g_{\mathrm{Euclidean}}$$

dilation is an isometry

- Conformal structure
- Conformal factor $\leftrightarrow$ distance to boundary.
- Curvature operator $\mathrm{Riem}[\breve{g}] = -\,\mathrm{id}$

Asymptotically hyperbolic manifold (M, g)

Conformal approach

- Defined by conformal compactification: $\overline{g} := \rho^2 g \in C^2(\overline{M})$
- Show $\mathrm{Riem}[g] \to -|d\rho|^2_{\overline{g}}\,\mathrm{id}$
- Require $|d\rho|_{\overline{g}} = 1$ at $\rho = 0$

Curvature approach

- Defined by $\mathrm{Riem}[g] \to -\mathrm{id}$
- Show one can attach conformal boundary
- Ask Eric Bahuaud...

Regularity at the conformal boundary

- Gap between two approaches
- What is necessary/sufficient for elliptic problems?

Example: Yamabe problem

Problem:

- Given (M^n, g), find ϕ such that $R[\phi^{4/(n-2)} g]$ is constant.
- Equivalent to

$$\Delta_g \phi - \frac{n-2}{4(n-1)} R[g]\phi = \frac{n(n-2)}{4}\phi^{(n+2)/(n-2)} \qquad \text{(Y)}$$

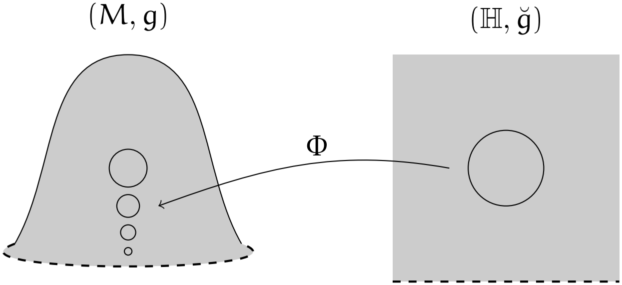
Tools needed

- Interior elliptic regularity
- Mapping properties of $\Delta_g - \frac{n-2}{4(n-1)} R[g]$
- Maximum principle / barriers

Why is boundary regularity needed?

Previously existing theory

- Approach of John M. Lee
  (see also Rafe Mazzeo, Lars Andersson & Piotr Chruściel)



- Suppose $\overline{g} = \rho^2 g \in C^{2,\alpha}(\overline{M})$ and $|d\rho|_{\overline{g}} \to 1$
- Then $\Phi^* g \approx \breve{g}$ and $\Delta_g \approx \Delta_{\breve{g}}$ uniformly near boundary:
  - Uniform elliptic estimates
  - Fredholm theory via parametrix construction

Yamabe problem in smooth setting

Works of Lars Andersson, Piotr Chruściel, Helmut Friedrich:

- Assume (M, g) is smoothly conformally compact: $\overline{g} = \rho^2 g \in C^\infty(\overline{M})$
- There exists solution ϕ to (Y); thus $g' = \phi^{4/(n-2)} g$ has constant scalar curvature...
- ... $g \in C^\infty(M)$, but $\rho^2 g'$ does *not* extend smoothly to $\overline{M}$.
- If $\dim(M) = 3$ there is a conformally-invariant obstruction.

It is desirable to

- solve Yamabe problem *within* a given regularity class, and
- better understand the role of the obstruction.

Weakly asymptotically hyperbolic metrics

Our approach:

- Define regularity classes of metrics with

$$\begin{aligned} &\text{high interior regularity:} && g \in C^{k,\alpha}(M), \\ &\text{low boundary regularity:} && \overline{g} = \rho^2 g \in W^{1,\infty}(\overline{M}) \subset C^{0,1}(\overline{M}) \end{aligned}$$

- Call such g *weakly asymptotically hyperbolic* if

$$|d\rho|_{\overline{g}} \to 1 \quad \Leftrightarrow \quad \mathrm{Riem}[g] \to -\mathrm{id}$$

- Show that this interior/boundary behavior is sufficient for:

$$\begin{matrix} g \approx \breve{g} \\ \Delta_g \approx \Delta_{\breve{g}} \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} \text{interior elliptic estimates} \\ \text{Fredholm theory} \end{matrix}$$

A few technical details

Hybrid Hölder spaces

- Fix smooth $\overline{h}$, $\overline{\nabla}$ on $\overline{M}$, set $h = \rho^{-2}\overline{h}$
- Interior spaces $C^{k,\alpha}(M)$ are equivalent to those given by h
- Require $\overline{g} = \rho^2 g$ to satisfy

$$g \in C^{k,\alpha}(M) \quad \Leftrightarrow \quad \overline{g} \in \rho^2 C^{k,\alpha}(M) \quad \Rightarrow \quad |\overline{g}|_{\overline{h}} \in L^\infty(\overline{M})$$

$$\overline{\nabla}\,\overline{g} \in \rho^3 C^{k-1,\alpha}(M) \quad \Rightarrow \quad |\overline{\nabla}\,\overline{g}|_{\overline{h}} \in L^\infty(\overline{M})$$

- Thus $\overline{g} \in C^{0,1}(\overline{M})$, but not necessarily in $C^1(\overline{M})$

Results, Part 1

Elliptic theory

- Extend Lee's Fredholm results to include *geometric elliptic operators* arising from weakly AH metrics

Yamabe problem

- If g is weakly AH, then we can solve (Y) and $g' = \phi^{4/(n-2)} g$ is also weakly AH.

Next up:

- Better understanding of obstruction
- Weakly AH solutions to Einstein constraint equations

Part 2:
The obstruction to higher boundary regularity
and
application to general relativity

Previous work (in dim 3)

Andersson–Chruściel–Friedrich:

- Suppose $\overline{g} = \rho^2 g \in C^\infty(\overline{M})$
- Can always find $\psi \in C^\infty(\overline{M})$ such that

$$R[\psi^4 g] = -6 + \rho^3 \mathcal{R}, \qquad \mathcal{R} \in C^\infty(\overline{M})$$

- Yamabe problem has $C^\infty(\overline{M})$ solution iff $\mathcal{R} \equiv 0$, equivalent to trace-free 2^{nd} FF of $\partial M \subset (\overline{M}, \overline{g})$ vanishing
- Obstruction related to "shear-free" condition in relativity

Our goals:

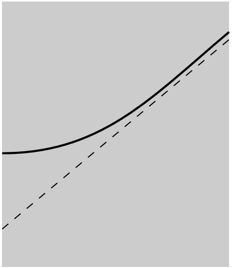
- Understand obstruction outside of smooth setting
- Construct shear-free initial data for Einstein equations

Asymptotically hyperbolic manifolds and spacetimes

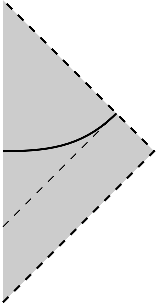
Minkowski spacetime $(\mathbb{R}^4, -dt^2 + dx^2 + dy^2 + dz^2)$

- Unit hyperboloid $-t^2 + x^2 + y^2 + z^2 = -1$

$\mathbb{R}^4/S^2$



Penrose compactification



Asymptotically hyperbolic manifolds

- Slices of asymptotically Minkowskian spacetimes
- Compatibility condition for C^0 spacetime compactification

Extrinsic curvature at conformal boundary

More regularity needed: Assume $\rho^{-4}\overline{\nabla}^2\overline{g} \in C^{k-2,\alpha}(M)$

- $\mathrm{Hess}(\rho)$ is well-defined at $\rho = 0 \rightsquigarrow 2^{\mathrm{nd}}$ FF of ∂M
- $\overline{g} \in C^{1,1}(\overline{M})$, but not necessarily $C^2(\overline{M})$.

Define tensor $\mathcal{H}_{\overline{g}}(\rho)$

- Conformally invariant version of $\mathrm{Hess}_{\overline{g}}(\rho)$
- Conformal extrinsic curvature of ∂M

Applications in weakly AH setting

- $\mathcal{H}_{\overline{g}}(\rho) \to 0 \leftrightarrow$ faster decay of curvature
- Conformally invariant description of shear-free condition

Application to general relativity

At each time, describe the "state" of system by

- metric g and 2-tensor K on M^3
- subject to constraint equations

$$R[g] - |K|_g^2 + (\mathrm{tr}_g K)^2 = \dots \qquad \text{and} \qquad \mathrm{div}_g K - d(\mathrm{tr}_g K) = \dots$$

Asymptotically hyperbolic data

- asymptotically Minkowskian $\rightsquigarrow$ describes isolated systems
- shear-free condition: trace-free $K = \mathcal{H}_{\overline{g}}(\rho)$ at ∂M

CMC-Conformal method:

Seek $g = \phi^4 \lambda$, $K = -g + \phi^{-2}(\mu + \mathcal{D}_\lambda W)$, ...

- must have

$$\begin{aligned} \Delta_\lambda \phi - \frac{1}{8} R[\lambda]\phi &= \frac{3}{4}\phi^5 - \frac{1}{8}|\mu + \mathcal{D}_\lambda W|^2_\lambda \phi^{-7} + \dots \\ \operatorname{div}_\lambda \mathcal{D}_\lambda W &= 0 + \dots \end{aligned} \qquad \text{(C)}$$

- Semi-decoupled elliptic system, generalizing (Y)

Andersson-Chruściel

- Do not expect C^2 conformal compactification

Work with Isenberg, Lee, Stavrov Allen (`arXiv:1506.06090`):

- Construct and classify weakly AH solutions to (C)
- Show shear-free condition can be built in to conformal method

Thank you!