The hyperboloidal initial value problem in general relativity

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Abstract

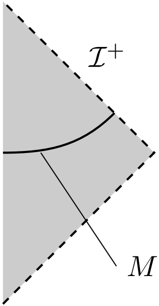
In asymptotically flat solutions to the Einstein vacuum equations, certain spacelike slices extending towards future null infinity have asymptotically hyperbolic geometry. In order for the spacetime to admit a regular conformal compactification, the geometry induced on such slices, which necessarily satisfies the Einstein constraint equations, must satisfy the *shear-free condition* along the conformal boundary. We refer to a spacelike manifold with such data as *hyperboloidal*.

The hyperboloidal initial value problem seeks to construct asymptotically flat spacetimes that arise from hyperboloidal initial data. A first step in addressing this initial value problem is constructing appropriate initial data. In this talk we provide motivation for considering the hyperboloidal initial value problem, give an overview of some technical issues that arise, and present some recent work regarding the existence of hyperboloidal initial data.

Asymptotically flat spacetimes

Minkowski spacetime

- Conformal structure:



- Foliate by hyperboloids M:

$$g = \breve{g}, \quad K = -\breve{g}$$

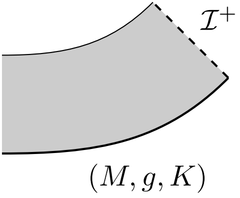
Asymptotically flat/simple spacetime

- Describes isolated gravitational system
- Conformal structure asymptotically Minkowskian
- Foliate by hyperboloidal slices M

$$\mathrm{Riem}[g] \to -\mathrm{id}, \quad K \to -g$$

  at conformal boundary
- Useful for numerics

Hyperboloidal initial value problem



- Hyperboloidal initial data (M, g, K)

$$\operatorname{div}_g K = d(\operatorname{tr}_g K),$$
$$\mathrm{R}[g] + (\operatorname{tr}_g K)^2 - |K|_g^2 = 0$$

- Find a spacetime $[0, T) \times M$

$$\mathcal{L}_\perp g = -2K,$$
$$\mathcal{L}_\perp K = \operatorname{Ric}[g] + (\operatorname{tr}_g K)K - 2K * K - \operatorname{Hess}_g N$$

where $\partial_t = N e_\perp + X$

Asymptotic conditions

Setup:

- $M = \mathrm{int}(\overline{M})$,
- fix a defining function $\rho \in C^\infty(\overline{M})$
- $\overline{g} = \rho^2 g, \quad K = \frac{\tau}{3} g + \Sigma$

Requirements

AH	$\mathrm{Riem}[g] \to -\mathrm{id} \quad \leftrightarrow \quad \lvert d\rho \rvert_{\overline{g}} \to 1$	
AF	$\tau \to -3$	due to **constraints**
SF	$\rho\Sigma \to \mathrm{Hess}_{\overline{g}}\, \rho - \frac{1}{3}(\Delta_{\overline{g}}\rho)\overline{g}$	due to **evolution**

Also

$$\rho N \to 1$$

$$X \to \mathrm{grad}_{\overline{g}}\, \rho$$

The SF condition

Focus on CMC setting: $K = -g + \Sigma$

- Preservation of CMC requires

$$\Delta_g N - (|\Sigma|_g^2 + 3)N = 0, \qquad \rho N \to 1$$

- Evolution equations imply:

$$\partial_t \overline{\Sigma} = \mathrm{Ric}[\overline{g}] - \frac{1}{3}\,\mathrm{R}[\overline{g}]\overline{g} + \dots$$
$$\dots + 2(\rho N)\frac{\mathrm{Hess}_{\overline{g}}\,\rho - \frac{1}{3}(\Delta_{\overline{g}}\rho)\overline{g} - \overline{\Sigma}}{\rho}$$

- Since $\rho N \to 1$, we need $\mathrm{Hess}_{\overline{g}}\,\rho - \frac{1}{3}(\Delta_{\overline{g}}\rho)\overline{g} - \overline{\Sigma} \to 0$
- Thus, regardless of which regularity class we consider, we need an appropriate version of SF

Smoothness at the conformal boundary

Friedrich (1986, 1987, 1988) If $\overline{g}, \tau, \rho\Sigma \in C^\infty(\overline{M})$, and asymptotic conditions, then there is a smooth solution to the initial value problem.

Andersson (2002) SF condition propagated under vacuum evolution in $C^\infty(\overline{M})$ setting

Andersson, Chruściel (1994) If $\overline{g}, \tau, \rho\Sigma \in C^\infty(\overline{M})$, but SF does not hold, then no smooth spacetime development admits a smooth conformal boundary.

Andersson, Chruściel (1996) "Most" initial data constructed by "usual" methods fails to extend smoothly to $\overline{M}$ and/or fails to satisfy SF

Boundary regularity for elliptic problems

- Smooth solutions to geometric elliptic problems in AH setting do not in general have C^∞ conformal structure.
- Polyhomogeneous expansions at $\rho = 0$ are typical: $u \sim \sum u_{kl} \rho^k (\log \rho)^l$

Andersson, Chruściel-Friedrich (1992) Yamabe problem

Andersson, Chruściel (1996) CMC conformal constraints

Bahuaud, Lee (2017) Poincaré-Einstein metrics

- What can go wrong? Resonance.

Example: Yamabe problem

- Suppose $\rho^2 g \in C^\infty(\overline{M})$.
- $\mathrm{R}[\phi^4 g] = -6$ requires

$$\Delta_g \phi = \frac{1}{8}\,\mathrm{R}[g]\phi + \frac{3}{4}\phi^5$$

- With $\phi = 1 + u$ this becomes

$$(\rho\partial_\rho + 1)(\rho\partial_\rho - 3)u + \cdots = \underbrace{\mathrm{R}[g] + 6}_{\mathcal{O}(\rho)} + Q(u).$$

- solution $u \in \rho C^\infty(M)$
- resonances at ρ^{-1} and ρ^3

  $\rightsquigarrow$ in general $u \in C^2(\overline{M})$, but not $C^3(\overline{M})$

CMC data I

Andersson, Chruściel (1996) use conformal method to construct solutions $(M, g, K = -g + \Sigma)$:

- smooth on M, polyhomogeneous on $\overline{M}$
- SF not necessarily satisfied

A—, Stavrov Allen (2017) construct perturbation $(M, g_\epsilon, K_\epsilon = -g_\epsilon + \Sigma_\epsilon)$ of A-C data:

- $\rho^2 g_\epsilon, \rho\Sigma_\epsilon \in C^\infty(\overline{M})$
- $g_\epsilon \to g$, $\Sigma_\epsilon \to \Sigma$ in $C^{k,\alpha}(M)$
- SF condition satisfied

- Friedrich's evolution theorems apply to perturbations
- SF condition not continuous in $C^{k,\alpha}(M)$ topology

Beyond smooth conformal structure?

> *"The overall picture that emerges ... is that the usual hypotheses of smoothness of Scri are overly restrictive."*
>
> Andersson–Chruściel (1994)

Desired:

- Solution theory for constraints that
  - closes within a regularity class
  - enforces SF
  - describes physically interesting systems
- Evolution theory for such data

CMC data II

Andersson, Chruściel (1994, 1996)

- closes in polyhomogeneous category
- can be used to construct SF solutions

A—, Isenberg, Lee, Stavrov Allen (2016)

- closes in "weakly AH" category

$$\mathscr{C}^{k,\alpha;m} \subset C^{k,\alpha}(M) \cap C^{m,1}(\overline{M}), \qquad m = 0, 1$$

- parametrizes all SF solutions in this category
- SF condition continuous in this topology

A—, Isenberg, Lee, Stavrov Allen (in preparation)

- gluing construction in weakly AH setting

Technical issues

Elliptic theory for $\rho^2 g \in C^{2+}(\overline{M})$

- Mazzeo: edge calculus
- Andersson; Lee: solutions in geometric Hölder, Sobolev spaces
- Sakovich: solutions in local Sobolev spaces

Elliptic theory for metrics of class $\mathscr{C}^{k,\alpha;m}$:

- A—, Isenberg, Lee, Stavrov Allen: solutions in geometric Hölder, Sobolev spaces

In the pipeline:

- Elliptic theory for Sobolev metrics

Outlook

Hyperboloidal data

- Lots of progress
- Technical improvements in progress

Evolution problem

- Appropriate regularity classes to be determined...
- Preservation of SF condition outside smooth category?

Stay tuned...